



BEST PROXIMITY POINT FOR CYCLIC CONTRACTION IN G –METRIC SPACE

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ABSTRACT

In this paper, we introduce the results of best proximity point in G -metric spaces for the cyclic contraction mapping with an example that illustrates the usability of the obtained results.

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INTRODUCTION

The existence and the convergence of best proximity points may be very applicable in non-linear analysis including optimization problems by considering the strong applications of fixed point theory. Mustafa [4] and Mustafa and Sims [7] in 2007 introduced the concept of generalized metric space or simply G -metric space as a generalization of the metric spaces. In this type of metric space, a non-negative real number is assigned to every triplet of elements. Many fixed-point results in this spaces have been appeared (for example, [6, 8, 5]). Recently, Samet *et.al.* [9] and Jleli- Samet [2] observed that some fixed-point theorems in the G -metric spaces can be concluded in the setting of a (quasi-) metric space. In fact, if the contraction condition of the fixed-point theorem on G -metric spaces can be reduced to two variables in-stead of three variables, then one can construct an equivalent fixed point theorem in the setting of a usual metric space. More precisely, in [2], the authors have noticed that the relation $d(x, y) = G(x, y, y)$ forms a quasi-metric space. Hence, if one can transform the contraction condition of existence results in G -metric space with this relation, then related fixed point results become the known fixed point results in the context of a quasi-metric space.

In this paper, we introduce the results of best proximity point for cyclic contraction in G -metric spaces that extends the result of Yadav *et.al.* [10] and improves other similar results.

PRELIMINARIES

In this section, we recollect some basic definitions and introduce some definitions in G -metric spaces.

DEFINITION 2.1 [7]: Let X be a nonempty set. Consider a function G defined by

$G : X \times X \times X \rightarrow R_+$ where R_+ is the set of positive real numbers, satisfying the following conditions:



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- (1) $G(x, y, z) = 0$ if and only if $x = y = z$;
- (2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
- (3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
- (4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables); and
- (5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G -metric on X and (X, G) is called a G -metric space.

DEFINITION 2.2 [7]: A G -metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

DEFINITION 2.3 [7]: Let (X, G) be a G -metric space. We say that the sequence $\{x_n\}$ is

- (1) a G -Cauchy sequence if, for any $\varepsilon > 0$, there is $M \in \mathbb{N}$ (the set of all positive integers) such that for all $n, m, l \geq M$, $G(x_n, x_m, x_l) < \varepsilon$; and
- (2) a G -convergent sequence to $x \in X$ if, for any $\varepsilon > 0$, there is $M \in \mathbb{N}$ (the set of all positive integers) such that for all $n, m \geq M$, $G(x, x_n, x_m) < \varepsilon$.

A G -metric space (X, G) is said to be complete if every G -Cauchy sequence in X is G -convergent in X .

PROPOSITION 2.1 [7]: Let (X, G) be a G -metric space. The following are equivalent:

- (1) $\{x_n\}$ is G -convergent to x ;
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$;
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$;
- (4) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow +\infty$.

PROPOSITION 2.2 [7]: Let (X, G) be a G -metric space. The following are equivalent:

- (1) the sequence $\{x_n\}$ is G -Cauchy;
- (2) $G(x_n, x_m, x_m) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Let A, B and C are non-empty subsets of a G -metric space (X, G) . A mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is called a cyclic mapping if $T(A) \subset B$, $T(B) \subset C$ and $T(C) \subset A$. Also, a point $z \in A \cup B \cup C$ is said to be Best Proximity point of T if $G(z, z, Tz) = G(A, B, C)$, where $G(A, B, C) = \{G(x, y, z) : x \in A, y \in B, z \in C\}$.



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Now, we deduce the condition in G -metric spaces from the condition given by Eldred and Veeramani [1] in metric spaces as follows:

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + (1 - \alpha)G(A, B, C) \tag{2.1}$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B, z \in C$ and $G(A, B, C)$ defined as above. Then, the mapping T satisfying the above condition is called Cyclic Contraction.

In this paper, we have established the following results that extend the existing theorem for best proximity point in metric spaces of Yadav *et.al* [10] to G -metric spaces and also give an example to illustrate our main theorem

MAIN RESULTS

THEOREM 3.1: Let A, B and C are non-empty closed subsets of a G -metric space (X, G) . Suppose that the mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ satisfy the condition (2.1) with $\alpha \in (0, 1)$, then there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+2}) = G(A, B, C)$.

PROOF: Suppose $x_0 \in A \cup B \cup C$ be given and consider a sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$.

From (2.1), we obtain

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+3}) &= G(Tx_n, Tx_{n+1}, Tx_{n+2}) \\ &\leq \alpha G(x_n, x_{n+1}, x_{n+2}) + (1 - \alpha)G(A, B, C). \end{aligned} \tag{3.1}$$

Again, we have

$$\begin{aligned} G(x_{n+2}, x_{n+3}, x_{n+4}) &= G(Tx_{n+1}, Tx_{n+2}, Tx_{n+3}) \\ &\leq \alpha G(x_{n+1}, x_{n+2}, x_{n+3}) + (1 - \alpha)G(A, B, C). \end{aligned} \tag{3.2}$$

Analogously, from (3.1) and (3.2), we conclude that

$$G(x_{n+2}, x_{n+3}, x_{n+4}) \leq \alpha^2 G(x_{n+1}, x_{n+2}, x_{n+3}) + (1 - \alpha^2)G(A, B, C).$$

Hence, inductively, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+2}) &\leq \alpha G(x_{n-1}, x_n, x_{n+1}) + (1 - \alpha)G(A, B, C) \\ &\leq \alpha^2 G(x_{n-2}, x_{n-1}, x_n) + (1 - \alpha^2)G(A, B, C) \\ &\leq \alpha^n G(x_0, x_1, x_2) + (1 - \alpha^n)G(A, B, C). \end{aligned}$$

Since $\alpha \in (0, 1)$, we have $\lim_{n \rightarrow \infty} \alpha^n = 0$, so the last inequality implies that



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$$\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+2}) = G(A, B, C).$$

Hence, the theorem is established.

THEOREM 3.2: Let A, B and C are non-empty closed subsets of a G -metric space (X, G) . Suppose that the mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ satisfy the condition (2.1) with $\alpha \in (0, 1)$. Assume that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in $A \cup B \cup C$. Then, x is a best proximity point of T .

PROOF: Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ that converges to some element x in $A \cap B$. Furthermore, by definition, we have

$$\begin{aligned} G(A, B, C) &\leq G(x, x_{2n_k-1}, x_{2n_k}) \\ &\leq G(x, x_{2n_k+1}, x_{2n_k+1}) + G(x_{2n_k+1}, x_{2n_k-1}, x_{2n_k}) \\ &\leq G(x, x_{2n_k+1}, x_{2n_k+1}) + G(A, B, C) \end{aligned}$$

Therefore, we have $G(x, x_{2n_k-1}, x_{2n_k}) \rightarrow G(A, B, C)$. Since T is cyclic contraction, it follows that $G(A, B, C) \leq G(x_{2n_k}, x_{2n_k+1}, Tx) = G(Tx_{2n_k-1}, Tx_{2n_k}, Tx)$

$$\leq \alpha G(x_{2n_k-1}, x_{2n_k}, x) + (1 - \alpha)G(A, B, C).$$

Taking $n \rightarrow \infty$ and using $G(x, x_{2n_k-1}, x_{2n_k}) \rightarrow G(A, B, C)$, we have

$$G(x, x, Tx) = G(A, B, C).$$

Hence x is the Best Proximity point of T .

We have the following example illustrating our main results.

EXAMPLE 3.1: Let $X = \mathbb{R}$, the set of real numbers and $G(x, y, z) = \begin{cases} 0, & x = y = z \\ \max\{x, y, z\}, & \text{otherwise} \end{cases}$

Then, (X, G) is a G -metric space. Let $T : A \cup B \cup C \rightarrow A \cup B \cup C$ be defined by

$$Tx = \begin{cases} x + 1, & x \in A \cup B - \{2\} \\ 2, & x = 2 \\ x - 2, & x \in C - \{2\} \end{cases}$$

where $A = [0, 1]$, $B = [1, 2]$ and $C = [2, 3]$.

Also, it is clear that $T(A) \subset B$, $T(B) \subset C$ and $T(C) \subset A$. Again, T satisfies the inequality (2.1) with G -metric defined as above. Hence, $x = 1$ and $x = 2$ are Best Proximity points of T .



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Again, let A, B and C are non-empty subsets of a G -metric space (X, G) . A mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is such that $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ satisfying the following condition:

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z) + \beta[G(x, x, Tx) + G(y, y, Ty) + G(z, z, Tz)] + \gamma G(A, B, C) \dots (3.1)$$

for all $x \in A, y \in B, z \in C$, where $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + 3\beta + \gamma < 1$. Then a mapping T is said to be generalized cyclic contraction.

With the notion of above generalized cyclic contraction, we have the following corollaries.

COROLLARY 3.1: Let A, B and C are non-empty closed subsets of a G -metric space (X, G) . Suppose that the mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ satisfy the condition (3.1), then there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+2}) = G(A, B, C)$.

COROLLARY 3.2: Let A, B and C are non-empty closed subsets of a G -metric space (X, G) . Suppose that the mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ satisfy the condition (3.1). Assume that the sequence $\{x_{2n}\}$ has a subsequence converging to some element x in $A \cup B \cup C$. Then, x is a best proximity point of T .

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