Kathmandu University
Journal of Science, Engineering and Technology

López-Bonilla *et. al.*, Vol. 13, No. II, December 2017, pp 92-97.

ORIGINAL RESEARCH ARTICLE OPEN ACCESS

IDENTITIES FOR HARMONIC NUMBERS AND BINOMIAL RELATIONS VIA LEGENDRE POLYNOMIALS

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Received 30 November, 2017; Revised 26 December, 2017

ABSTRACT
We employ the orthonormality of the Legendre polynomials to deduce binomial identities. The harmonic numbers H_n are connected with the derivatives of binomial coefficients, this fact allows to deduce identities involving the H_n.

Keywords: Legendre polynomials, Schmied's formula, Harmonic and Stirling numbers, Binomial coefficients

INTRODUCTION
Legendre polynomials are given by [1-3]:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, \qquad n \geq 0, \tag{1}$$

with the property $P_n(1) = 1 \;\; \forall\, n$, then from (1):

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = 2^n \,. \tag{2}$$

Besides, we have the orthonormality relation:

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{mn} \,, \tag{3}$$

that is:

$$\int_{-1}^{1} x^m\, P_n(x)\, dx = 0, \qquad m < n, \tag{4}$$

$$\int_{-1}^{1} x^n\, P_n(x)\, dx = \frac{2^{n+1}}{(2n+1)\binom{2n}{n}} = \frac{2^{n+1}\,(n!)^2}{(2n+1)!} = \frac{2\,(n!)}{(2n+1)!!} \,. \tag{5}$$

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If $m-n=$ odd integer then $x^m P_n(x)$ is an odd function, hence:

$$\int_{-1}^{1} x^m \; P_n(x)\, dx = 0, \qquad m-n = 1, 3, 5, \ldots \tag{6}$$

In Sec. 2 we employ (3) and the Schmied's formula [4] to obtain the expression:

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{2^{n+1}}{m+1} \frac{\binom{\frac{m+n}{2}}{n}}{\binom{m+n+1}{n}}, \qquad m-n = 0, 2, 4, \ldots \tag{7}$$

which implies (5) if $m = n$. We also use (1), (4) and (7) to deduce binomial identities similar to (2).

It is well known the property:

$$\frac{d}{dx}\binom{x+m}{n} = \binom{x+m}{n} \sum_{j=1}^{n} \frac{1}{j+x+m-n}, \tag{8}$$

in particular:

$$[\frac{d}{dx}\binom{x+m}{n}]_{x=n-m} = H_n, \qquad [\frac{d}{dx}\binom{x}{n}]_{x=-1} = (-1)^{n+1} H_n, \tag{9}$$

for the harmonic numbers [5]:

$$H_n = \sum_{r=1}^{n} \frac{1}{r}, \qquad n \geq 1, \qquad H_0 = 0. \tag{10}$$

In Sec. 3 we employ (8) and (9) to deduce identities involving the quantities (10).

Schmied's Formula

In [4] we find the following relation of Schmied (2005):

$$x^m = \sum_{l=m,m-2,\ldots} \frac{m!\,(2l+1)}{2^{\frac{m-l}{2}} \left(\frac{m-l}{2}\right)! \,(m+l+1)!!} P_l(x), \tag{11}$$

where, we can multiply by $P_n(x)$, to integrate in [-1, 1], and apply (3), to obtain (7) for

$$m-n = 0, 2, 4, \ldots$$

Now we use (1) into (4) and (7) to deduce the result:

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$$\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n}{k}\binom{2n-2k}{n} \frac{(-1)^k}{m+n+1-2k} = \begin{cases} 0, & m < n \\ \dfrac{4^n \binom{\frac{m+n}{2}}{n}}{(m+1)\binom{m+n+1}{n}}, & m > n \end{cases} \tag{12}$$

for $m+n = 2, 4, 6, \ldots$ Similarly:

$$\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n}{k}\binom{2n-2k}{n} \frac{(-1)^k}{2n+1-2k} = \frac{4^n\,(n!)^2}{(2n+1)!}, \qquad n = 0, 1, 2, \ldots \tag{13}$$

Remark- From [6] we have the formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{z+ky}{n} = (-y)^n, \qquad y \neq 0, \tag{14}$$

which, for $y = -2$ and $z = 2n$ is equivalent to (2) because $\binom{2n-2k}{n} = 0$ for $k > \left\lfloor\frac{n}{2}\right\rfloor$.

Harmonic Numbers

We have the expression [6]:

$$x^n = \sum_{j=0}^{n} j!\binom{x}{j} S_n^{[j]}, \tag{15}$$

where, $S_n^{[j]}$ are Stirling numbers of the second kind [6- 8]. Now (9) and $\left[\frac{d}{dx}(15)\right]_{x=-1}$ imply:

$$\sum_{j=1}^{n} (-1)^j\, j!\, H_j\, S_n^{[j]} = n\,(-1)^n, \qquad n \geq 1. \tag{16}$$

We can verify (16), in fact [6, 9]:

$$H_j = \frac{(-1)^j}{j!} \sum_{q=1}^{j} (-1)^q\, q\, S_j^{(q)}, \tag{17}$$

for the Stirling numbers of the first kind $S_n^{(m)}$, then:

$$\sum_{j=1}^{n} (-1)^j\, j!\, H_j\, S_n^{[j]} = \sum_{q=1}^{n} (-1)^q\, q \sum_{j=q}^{n} S_n^{[j]}\, S_j^{(q)} = (-1)^n\, n,$$

by the orthonormality of the Stirling numbers [6]; hence (16) and (17) are reciprocal relations.

Lanczos [10] used the binomial expansion of Gregory-Newton to obtain the identity:

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$$\sum_{k=0}^{n} \binom{x}{k}\binom{n}{k} \frac{1}{(k+1)_m} = \frac{1}{(n+1)_m} \binom{x+m+n}{n}, \tag{18}$$

where, $(k+1)_m = \frac{(k+m)!}{k!}$; then (9) and $[\frac{d}{dx}(18)]_{x=-1}$ allow to deduce the formula:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k+1)_m} \binom{n}{k} H_k = \frac{1}{(m-1)!\,(m+n)} (H_{m+n-1} - H_{m-1}), \qquad m \geq 1. \tag{19}$$

We have the following expression of Graham-Knuth [11]:

$$\sum_{k=0}^{n} \binom{x+k}{k} = \left(1 + \frac{n}{x+1}\right)\binom{x+n}{n}, \qquad n \geq 0, \tag{20}$$

therefore, (9) and $[\frac{d}{dx}(20)]_{x=0}$ imply the property [12]:

$$\sum_{k=0}^{n} H_k = (n+1)\, H_n - n, \qquad n = 0, 1, 2, \dots, \tag{21}$$

which is a particular case of the identity [9, 11-15]:

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right), \tag{22}$$

for $m = 0$.

In [10] we find the relation:

$$\sum_{k=1}^{n} (-1)^k \binom{x}{k}\, k = (-1)^n\, x \binom{x-2}{n-1}, \qquad n \geq 1, \tag{23}$$

thus, (9) and $[\frac{d}{dx}(23)]_{x=-1}$ generate the result [16]:

$$\sum_{k=1}^{n} k\, H_k = \binom{n+1}{2}\left(H_{n+1} - \frac{1}{2}\right), \tag{24}$$

which is deductible from [6, 13]:

$$\sum_{k=1}^{n} k^m H_k = \sum_{j=1}^{m} \binom{n+1}{j+1}\left(H_{n+1} - \frac{1}{j+1}\right) j!\; S_m^{[j]}, \tag{25}$$

for $m = 1$.

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We know the expression:

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\binom{x+k}{k}}=\sum_{k=1}^{n}\frac{1}{x+k}\ ,\tag{26}$$

then, $[\frac{d}{dx}(26)\ ]_{x=0}$ and (9) allow to obtain the identity:

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k}\,H_k=\sum_{k=1}^{n}\frac{1}{k^2},\tag{27}$$

which can be verified directly via the relation:

$$H_k=\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^{j+1}}{j},\tag{28}$$

consequence from (26) for $x=0$.

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