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SOME FIXED POINT RESULTS BY ALTERING DISTANCES BETWEEN POINTS

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ABSTRACT
The theory of fixed point is a very extensive field, which has various applications. This paper is the survey work on some fixed point theorems by altering distances between points in metric space.

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INTRODUCTION

Theorems concerning the existence and properties of fixed points are known as *fixed point theorem*. The theory of fixed points has become an important tool in non-linear functional analysis since 1930. It is used widely in applied mathematics. The existence and types of solution always help to give geometrical interpretation, to discuss the behaviour and to check stability of the concern system. This is the basis for the modelling of a system. The fixed point theorems related to altering distances between points in complete metric space have been obtained initially by D. Delbosco in 1967, F. Skof in 1977, M.S. Khan, M. Swaleh and S. Sessa in 1984. The paper of Pant *et al.* [26] deals with the survey work on the history of fixed point theorems.The purpose of the present paper is to study the common fixed point theorem by altering distances between the points through different types of mappings, like contraction, non-expansive, sequence of mappings, Fuzzy mappings.

DEFINITION 1.1. Let X be a set and T a map from X to X. A *fixed point* of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x$.

THEOREM 1.1. (Banach Contraction Principle): Let (X, d) be a complete metric space and $F: X \to X$ be a contraction mapping, then F has a unique fixed point.

DEFINITION 1.2. The c*ontrol function* ψ is defined as $\psi: \mathfrak{R}_+ \to \mathfrak{R}_+$ which is continuous at zero, monotonically increasing, $\psi(2t) \leq 2\,\psi(t)$ and $\psi(t) = 0$ if and only if $t = 0$.

DEFINITION 1.3. Two self mappings A and S of a metric space (X, d) are called ψ-*compatible* if $\lim_{n\to\infty} \psi(d(ASx_n, SAx_n)) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

DEFINITION 1.4. Two self mappings A and S of a metric space (X, d) are said to be *reciprocally continuous* in X if, $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

DEFINITION 1.5. Maps A and S are *pointwise R-weakly commuting* on X given if $x \in X$, there exists $R > 0$ such that $d(ASx, SAx) \leq R\, d(Ax, Sx)$. It is noted that weak commutativity of a pair of selfmaps implies their R-weak commutativity and reverse implication is true only when $R \leq 1$ [39].

DEFINITION 1.6. Let Ψ_n denote the set of all functions $\psi : [0, \infty)^n \to [0, \infty)$ such that
(i) ψ is continuous, and (ii) $\psi(t_1, t_2, \ldots, t_n) = 0$ if and only if $t_1 = t_2 = \ldots = t_n = 0$,
Then, the functions in Ψ_n are called *generalized altering distance function*.

2. FIXED POINT THEOREMS BY ALTERING DISTANCES BETWEEN POINTS FOR PAIR OF SELF MAPPINGS

In 1976-1977, D. Delbosco [7] and in 1977, F. Skof [38] have established fixed point theorems for self maps of complete metric spaces by altering the distances between the points with the use of a function $\varphi: \mathfrak{R}_+ \to \mathfrak{R}_+$ satisfying the following properties:

1). φ is continuous and strictly increasing in $\mathfrak{R}_+$; (2). $\varphi(t) = 0$ if and only if $t = 0$;
3). $\varphi(t) \geq Mt^{\mu}$ for every $t > 0$, where $M > 0$, $\mu > 0$ are constant.

The set of all such functions φ is denoted by Φ.

In 1977, F. Skof proved the following theorem with φ defined above for single self mapping.

THEOREM 2.1[38]: Let T be a selfmap of a complete metric space (X, d) and $\varphi \in \Phi$ such that for every x, y in X, $\varphi(d(Tx, Ty)) < a.\varphi(d(x, y)) + b.\varphi(d(x, Tx)) + c.\varphi(d(y, Ty))$, (2.1)
where, $0 \leq a + b + c < 1$. Then T has a unique fixed point.

Also, Delbosco [7] considered functions $\varphi \in \Phi$ such that $\varphi(t) = t^n$, for $n \in N$ and for every $t \geq 0$.

In 1984, Khan *et al.* proved the following theorem without assuming the continuity of mapping and generalizing the result of Skof.

THEOREM 2.2 [19]: Let (X, d) be a complete metric space, T a selfmap of X, and $\varphi: \mathfrak{R}_+ \to \mathfrak{R}_+$ an increasing, continuous function satisfying property (2) of Theorem 2.1. Furthermore, let a, b, c be three decreasing functions from $\mathfrak{R}_+ \setminus \{0\}$ into [0, 1] such that
$a(t) + 2b(t) + c(t) < 1$ for every $t > 0$. Suppose that T satisfies the following condition:

$$\varphi(d(Tx, Ty)) \leq a(d(x, y)).\ \varphi((d(x, y) + b(d(x, y)).\{\varphi(d(x, Tx)) + \varphi(d(y, Ty)))$$
$$+ c(d(x, y)).\min\{\varphi(d(x, Ty)), \varphi(d(y, Tx))\}, \quad (2.2)$$

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

It is noted that in Theorem 2.2, φ is not necessarily a metric: For example, consider $\varphi = t^2$.
Also, by the symmetry of metric d, we may assume b = c in Theorem 2.1. Moreover, if we assume c = 0 in Theorem 2.2 and take a, b as constants, we obtain the result of Skof. Again, in Theorem 2.2, if we assume c = 0 and $\varphi(t) = t$ for every $t \geq 0$, we obtain the following condition

$$d(Tx,Ty) \leq a(d(x, y)).d(x, y) + b(d(x, y)).\{d(x, Tx) + d(y, Ty)\}. \quad (2.3)$$

If we assume b = c = a in Theorem 2.2, we get the following result.

THEOREM 2.3: Let (X, d) be a complete metric space, T a selfmap of X and $\varphi : \mathfrak{R}_+ \to \mathfrak{R}_+$ be an increasing, continuous function for which property (2) of Theorem 2.1 holds. Let a be a decreasing function from $\Re_+ \setminus \{0\}$ into [0, l] such that $\varphi(d(Tx, Ty)) \leq a(d(x, y)).\ \varphi(d(x, y))$, (2.4)

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In 1976, B. Fisher established the following theorem in compact metric space.

THEOREM 2.4 [8]: Let T be a continuous selfmap of a compact metric space (X, d) such that for all x, y in X, $d(Tx, Ty) < [d(x, Tx) + d(y, Ty)]/2$. Then T has a unique fixed point.

In 1984, Khan *et al.* generalized the above Theorem 2.4 as follows:

THEOREM 2.5 [19]: Let T be a continuous selfmap of a metric space (X, d) such that for some $x \in X$ the sequence $\{T^n x_o\}$ has a cluster point $z \in X$. Let there exists a continuous function $\varphi : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$ satisfying property (2) of Theorem 2.1. Furthermore, for all distinct x, y in X the inequality

$$\varphi(d(Tx, Ty)) < c\, \varphi(d(x, y)) + (1 - c)\, [\, \varphi(d(x, Tx) + \varphi(d(y, Ty))]/2 \qquad (2.5)$$

holds, where $0 \leq c \leq 1$. Then z is the unique fixed point of T.

Let (X, d) denotes the metric space, $\Re_+$ the set of all non-negative real numbers, $\aleph$ the set of all natural numbers and Φ the set of all continuous self mappings ϕ of $\Re_+$ satisfying $\phi(t) = 0$ if and only if $t = 0$. With this notation, in Sastry *et al.* in 1999 established the following theorem on the orbit.

THEOREM 2.6 [36]: Let T be a selfmap on a metric space (X, d). Suppose there exists a point x_0 in X such that the orbit $O(x_0) = (T^n x_0 : n = 0, 1, 2, \ldots)$ has a cluster point z in X. If T is arbitrally continuous at z and Tz and there exists a $\phi \in \Phi$ such that $\phi(d(Tx, Ty)) < \phi(d(x, y))$ (2.6)

for each $x, y = Tx \in \overline{O(x_0)}$; $x \neq y$ then z is a fixed point of T.

THEOREM 2.7[36] : Let T be a continuous selfmaps of a metric space (X, d) such that for some x_0 in X, the sequence $\{T^n x_0\}$ has a cluster point in X; and there exists $\phi \in \Phi$ such that

$$\phi(d(Tx, Ty)) < \max\{\phi(d(x, y)), \phi(d(x, Tz), \phi(d(y, Ty))\} \qquad (2.7)$$

for all distinct x, y in X. Then z (of Theorem 2.5) is the unique fixed point of T.

In 2001, G.V.R. Babu and S. Ismail proved the following theorem in complete metric space.

THEOREM 2.8[1]: Let (X, d) be a complete metric space, and T a selfmap of X. Assume that T satisfies the following inequality: There is a $k \in [0, 1)$ and $\psi \in \Psi$ such that

$$\psi(d(Tx, Ty) \leq k \max\{\psi(d(x, y)), \psi(d(x, Tx)), \psi(d(y, Ty)), [\psi(d(x, Ty) + \psi(d(y, Tx)]/2\}$$

For all $x, y \in X$. For any x_0 in X, define $x_n = T^n x_o$, n = 1, 2, … Then, $\{x_n\}_{n=1}^{\infty}$ is Cauchy, $\lim_{n\to\infty} x_n$ exists, say z is the unique fixed point of T in X.

In 2000, Sastry *et al.* proved the following theorem in complete metric space.

THEOREM 2.9[37]: Let (X, d) be a complete metric space, T a selfmap of X. Assume that T satisfies the following inequality: there is a $k \in (0, 1)$ and φ in Φ such that

$$\varphi(d(Tx, Ty)) \leq k \max\{\varphi(d(x, y)), \varphi(d(x, Tx)), \varphi(d(y, Ty))\}, \text{ for all } x, y \text{ in X}.$$

Then T has a unique fixed point in X.

In 1999, K.P.R. Sasty and G.V.R. Babu proved the following theorems in a metric space.

THEOREM 2.10.[35]: Let S and T be selfmaps of a metric space (X, d). For $x_0 \in X$, define the sequence $\{x_n\}$ by $x_{2n+2} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$. Suppose either (A) $\{x_{2n}\}$ has a cluster point z

in X; and S, ST are continuous at z or (B)$\{x_{2n+1}\}$ has a cluster point z in X; and T, ST are continuous at z.
Assume that there exists a $\phi \in \Phi$ such that $\phi(d(Sx, Ty)) < \phi(d(x, y))$
(2.8)
for each distinct $x, y \in \{\overline{x_n}\}$ satisfying either $x = Ty$ or $y = Sx$. Then, (1) either S or T has a fixed point in $\{x_n\}$ or (2) z is a common fixed point of S and T.

THEOREM 2.11. [35]: Let (X, d) be a bounded complete metric space and S and T be selfmaps of X such that ST = TS. Further, assume that S and T satisfy the following inequality:
there exists $k \in (0, 1)$ and $\varphi \in \Phi$ such that $\varphi((Sx, Ty)) \leq k.\max\{\varphi(d(x, y)), \varphi(d(x, Sx)), \varphi(d(y, Ty))\}$,
for all x, y in X. Then one of S and T (and hence both) have a unique common fixed point in X.

3. FIXED POINT THEOREMS BY ALTERING DISTANCES FOR PAIRS OF SELF MAPPINGS

The most general common fixed theorems for four mappings, say A, B, S and T of a metric space
(X, d) use either Banach type contractive condition of the form
$d(Ax, By) \leq h\, m(x, y), \quad 0 \leq h < 1$ where
(3.1)
$m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}$,
or, a Meir-Keeler type (ε, δ) contractive condition [20] of the form: given $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon$,
(3.2)
or, a ϕ contractive condition [2] of the form: $d(Ax, By) \leq \phi(m(x, y))$,
(3.3)
involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.
The weak form of the contractive condition (3.2) is of the form
$\varepsilon < m(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon$.
(3.4)
Clearly, condition (3.1) is a special case of both conditions (3.2) and (3.3).
A ϕ-contractive condition does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3.3), the following condition on ϕ have been used by various authors.

- (I) $\phi(t)$ is non decreasing and $t / (t - f(t))$ is non drecreasing (Carbone *et.al.*[6])
- (II) $\phi(t)$ is non decreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$. (Jachymski [12])
- (III) ϕ is upper semi continuous (Boyd and Wong[2], Jachymski [12], Maiti and Pal [21], Pant[24]) or equivalently,
- (IV) ϕ is non decreasing and continuous from right (Park and Rhoades [30])

It is now known (e.g. Jachymski [12], Pant *et.al.* [23]) that if any of the conditions (I), (II), (III), or (IV) is assumed on ϕ, then a ϕ - contractive condition (3.3) implies an analogous (ε, δ)- contractive condition (3.2) and both the contractive conditions hold simultaneously. In 1998, R.P. Pant established the following fixed point theorem for compatible pair of reciprocally continuous maps.

THEOREM 3.1. [**22**]: Let (A, S) and (B, T) be point wise R-weakly commuting pairs of self mappings of a complete metric space (X, d) such that
(i) $AX \subset TX$, $BX \subset SX$, (ii) $d(Ax, By) \leq h\, M(x, y)$, $0 \leq h < 1$, for $x, y \in X$ where
$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}$.
Suppose that (A, S) or (B, T) is compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.
In 2000, Sastry *et al.* established the following fixed point theorems for ψ-compatible pair of self mapping extending above Theorem 3.1.

THEOREM 3.2. [37]: Let (A, S) and (B, T) be weakly commuting pairs of self maps of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying (i) $AX \subset TX$, $BX \subset SX$ and
(ii) there exists h in [0, 1) such that $\psi(d(Ax, By)) \leq h\, M_\psi(x, y)$, where
$M_\psi(x, y) = max\,\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$,
for all x, y in X. Suppose that (A, S) or (B, T) is a ψ-compatible of reciprocally continuous mappings. Then, A, B, S and T have a unique common fixed point.

THEOREM 3.3. [37]: Let (A, S) and (B, T) be weakly commuting pairs of self maps of a complete metric space (X, d) and ψ be as in as in definition (1.2) satisfying (i) $AX \subset TX$, $BX \subset SX$ and
(ii) there exists h in [0, 1) such that $\psi(d(Ax, By)) \leq h\, M_\psi(x, y)$, where
$M_\psi(x, y) = max\,\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, for all x, y in X. Suppose that A and S are ψ-compatible and S is continuous. Then, A, B, S and T have a unique common fixed point.
In 2003, Pant *et al.* established the following fixed point theorems for ψ-compatible pair of reciprocally continuous self mappings.

THEOREM 3.4.[27]: Let (A, S) and (B, T) be weakly commuting pairs of self maps of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying (i) $AX \subset TX$, $BX \subset SX$ and
(ii) $\psi(d(Ax, By)) \leq \phi(M_\psi(x, y))$, for all x, y in X whenever $M_\psi(x, y) > 0$, where
$M_\psi(x, y) = max\,\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$,
and $\phi:\mathfrak{R}_+\to\mathfrak{R}_+$ be an upper semi continuous function such that $\phi(t) < t$ for each $t > 0$.
Suppose that (A, S) and (B, T) is ψ-compatible pairs of reciprocally continuous mappings. Then, A, B, S and T have a unique common fixed point.

THEOREM 3.5.[27]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying (i) $AX \subset TX$, $BX \subset SX$ and (ii) $\psi(d(Ax, By)) \leq \phi(M_\psi(x, y))$, for all x, y in X whenever $M_\psi(x, y) > 0$,where
$M_\psi(x, y) = max\,\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, and $\phi: \mathfrak{R}_+ \to \mathfrak{R}_+$ be an upper semi continuous function such that $\phi(t) < t$ for each $t > 0$. Suppose that A and S are ψ-compatible and A is continuous mapping. Then, A, B, S and T have a unique common fixed point.
In 2003, Pant *et al.* established the following fixed point theorems for ψ-compatible pair of self mappings.

THEOREM 3.6.[28]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and function ψ be as in definition (1.2) satisfying: (i) $AX \subset TX$,
$BX \subset SX$ and (ii) there exists h in [0, 1) such that $\psi(d(Ax, By)) \leq h\, M_\psi(x, y)$, where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, for all x, y in X. Suppose that A and S are ψ-compatible and A is continuous. Then A, B, S and T have a unique common fixed point.
Pant *et al.* (2003) established the following fixed point theorems for ψ-compatible pair of self mapping.

THEOREM 3.7.[29]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and function ψ be as in definition (1.2) satisfying (i) $AX \subset TX$,
$BX \subset SX$ and (ii) there exists h in [0, 1) such that $\psi(d(Ax, By)) \leq h\, M_\psi(x, y)$, where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, for all x, y in X. Then the continuity of one of the mappings in ψ-compatible pair (A, S) on X implies their reciprocal continuity.

THEOREM 3.8.[29]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and function ψ be as in definition (1.2) satisfying (i) $AX \subset TX$,
$BX \subset SX$ and (ii) there exists h in [0, 1) such that $\psi(d(Ax, By)) \leq h\, M_\psi(x, y)$, where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, for all x, y in X. Let (A, S) and (B, T) be ψ-compatible. If S or T is continuous Then A, B, S and T have a unique common fixed point.
In 2004, K. Jha and R.P. Pant established the following fixed point theorems for ψ-compatible pair of reciprocally continuous self mappings.

THEOREM 3.9[15]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying (i) $AX \subset TX$, $BX \subset SX$ and
(ii) $\psi(d(Ax, By)) \leq \phi(M_\psi(x, y))$, for all x, y in x whenever $M_\psi(x, y) > 0$, where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, and
$\phi : \mathfrak{R}_+ \to \mathfrak{R}_+$ be an upper semi continuous function such that $\phi(t) < t$ for each $t > 0$. Suppose that
(A, S) and (B, T) are ψ-compatible pairs of reciprocally continuous mappings. Then, A, B, S and T have a unique fixed point.

THEOREM 3.10.[15]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and function ψ be as in definition (1.2) satisfying. (i) $AX \subset TX$,
$BX \subset SX$ and (ii) $\psi(d(Ax, By)) \leq \phi(M_\psi(x, y))$, for all x, y in X whenever $M_\psi(x, y) > 0$, where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, and $\phi : \mathfrak{R}_+ \to \mathfrak{R}_+$ be an upper semi continuous function such that $\phi(t) < t$ for each $t > 0$.

Suppose that A and S are ψ-compatible and A is continuous mapping. Then, A, B, S and T have a unique fixed point. The proof is similar when the pair (A, S) is assumed ψ-compatible and S is continuous. Moreover, we can get the same result when the (B, T) is assumed ψ-compatible and either T or B is assumed continuous.

EXAMPLE 3.3. Let X = [0, 1] with the Euclidean metric d. Define A = B and S = T : X $\to$ X by the rule A0 = ½, Ax = x/4 for $0 < x \leq 1$ and S0 =1, Sx = x/2 for $0 < x \leq 1$. Then A and S are weakly commuting mappings and hence they are ψ-compatible, with ψ being an identity mapping. Also, A and S are not continuous and they do not have common fixed point.
In 2006, Jha *et al.* established the following fixed point theorems for ψ-compatible pair of reciprocally continuous self mappings.

THEOREM 3.11.[16]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying : (i) $AX \subset TX$, $BX \subset SX$ and
(ii) $\psi(d(Ax, By)) < \phi(M_\psi(x, y))$, for all x, y in X whenever $M_\psi(x, y) > 0$, where
$M_\psi(x, y) = max\ \{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty))\}$, and $\phi: \mathfrak{R}_+ \to \mathfrak{R}_+$ be a non decreasing function such that $\phi(t) < t$ for each $t > 0$ and $\lim_{n\to\infty}\phi^n(t) = 0$, $\phi^n(t)$ being composition of $\phi(t)$ with itself n-times. Suppose that (A, S) is ψ-compatible pair and A is continuous mapping. Then A, B, S and T have a unique common fixed point.

THEOREM 3.12.[16]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying: (i) $AX \subset TX$, $BX \subset SX$ and
(ii) $\psi(d(Ax, By)) < \phi(M_\psi(x, y))$, for all x, y in X, whenever $M_\psi(x, y) > 0$, where
$M_\psi(x, y) = max\ \{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty))\}$, and $\phi: \mathfrak{R}_+ \to \mathfrak{R}_+$ be a non decreasing function such that $\phi(t) < t$ for each $t > 0$ and $\lim_{n\to\infty}\phi^n(t) = 0$, $\phi^n(t)$ being the composition of $\phi(t)$ with itself n-times. Suppose that (A, S) and (B, T) are ψ-compatible pairs of reciprocally continuous mappings. Then, A, B, S and T have a unique common fixed point.
In 2007, Rao *et al.* established the following fixed point theorems for ψ-compatible pair of self mappings using continuity condition on self maps.

THEOREM 3.13.[33]: Let P, Q, S and T be self mapping of a complete metric space (X, d) satisfying:
(i) $\phi_1(d(Px, Qy)) \leq \psi_1(d(Sx, Ty), d(Sx, Px), d(Ty, Qy), [d(Sx, Qy) + d(Ty, Px)]/2)$
$- \psi_2(d(Sx, Ty), d(Sx, Px), d(Ty, Qy), [d(Sx, Qy) + d(Ty, Px)]/2)$ (3.5)
for all $x, y \in X$, where $\psi_1, \psi_2 \in \mathbf{\Phi_4}$ and $\phi_1(x) = \psi_1(x, x, x, x)$ for all $x \in [0, \infty)$,
(ii) either S and T or P and T or Q and S are continuous, (iii) (P, S) and (Q, T) are compatible pairs of type (B), (iv) $PT(X) \cup QS(X) \subset T(X)$ and ST = TS. Then, P, Q, S and T have a unique common fixed point in X.
In 2007, Jha *et al.* established the following fixed point theorems for ψ-compatible pair of self mappings using Meir-Keeler type contractive condition.
THEOREM 3.14.[17]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying: (i) $AX \subset TX$, $BX \subset SX$ and (ii) given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all x, y in X

$$\varepsilon \leq M_\psi(x, y) < \varepsilon + \delta \Rightarrow \psi(d(Ax, By)) < \varepsilon, \text{ and (iii) } \psi(d(Ax, By)) \leq \phi(M_\psi(x, y)),$$

for all x and y in X whenever $M_\psi(x, y) > 0$, where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, and $\phi: \mathfrak{R}_+ \to \mathfrak{R}_+$ is such that ϕ is non decreasing and $\phi(t) < t$ for each $t > 0$. Suppose that (A, S) and
(B, T) are ψ-compatible pairs of reciprocally continuous mappings. Then, A, B, S and T have a unique common fixed points.

EXAMPLE 3.5 [17]: Let X = [2, 20] with the usual metric d on X. Define A, B, S, T : X $\to$ X
by Ax = 2 for each x; Bx = 2 if x = 2 or $x > 3$, Bx = 8 - x if $2 < x \le 3$; Sx = x if $x \le 6$ if $x > 6$;
Tx = 2 if $x = 2$ or $x > 3$, Tx = 8+ x if $2 < x \le 3$. Then, AX $\subset$ TX, BX $\subset$ SX, (A, S) and (B, T) are weakly commuting compatible pairs of reciprocally continuous mapping. The mappings A, B, S, T have a unique common fixed point $x = 2$.

In 2007, Jha *et al.* established the following fixed point theorems for ψ-compatible pair of self mappings using Meir-Keeler type contractive condition.

THEOREM 3.15.[17]: Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.2) satisfying: (i) AX$\subset$ TX, BX $\subset$ SX and
(ii) given $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \le M_\psi(x, y) < \varepsilon + \delta \Rightarrow \psi(d(Ax, By)) < \varepsilon$,
(iii) $\psi(d(Ax, By)) \le \phi((M_\psi(x, y))$, for all x and y in X, whenever $M_\psi(x, y) > 0$,where
$M_\psi(x, y) = max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), [\psi(d(Sx, By)) + \psi(d(Ax, Ty))]/2\}$, and $\phi: \mathfrak{R}_+ \to \mathfrak{R}_+$ is such that ϕ is non decreasing and $\phi(t) < t$ for each $t > 0$. Suppose that A, and S are ψ-compatible and S is continuous mappings. Then, A, B, S and T have a unique common fixed points.

In 2008, Rao *et al.* established the following fixed point theorems for ψ-compatible pair of self mappings using continuity condition, generalizing the result of B. Chaudhary [3].

THEOREM 3.16.[34]: Let (X, d) be a complete metric space and f, g, S, T : X $\to$ X be such that
(i) $\phi_1(d(fx, gy)) \le \psi_1(d(Sx, Ty), d(Sx, fy), d(Ty, gy), [d(Sx, gy) + d(Ty, fx)]/2)$,
$- \psi_2(d(Sx,Ty), d(Sx, fx), d(Ty, gy), [d(Sx, gy) + d(Ty, fx)]/2)$
for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_4$ and $\phi_1(x) = \psi_1(x, x, x, x)$ for all $x \in [0, \infty)$, and
(ii) One of mappings f, g, S and T is continuous, (iii) (f, S) and (g, T) are semi compatible pairs, and (iv) f(X) $\subseteq$ T(X), g(X) $\subseteq$ S(X). Then f, g, S and T have a unique common fixed point in X.

4. FIXED POINT THEOREMS BY ALTERING DISTANCES BETWEEN POINTS FOR SEQUENCE OF MAPPINGS

In 1974, Iseki established the following fixed point theorems for sequence of self maps in complete metric space.

THEOREM 4.3.[11]: Let (X, d) be a complete metric space, and $\{T_n\}_{n=1}^{\infty}$ be a sequence of selfmap of X. Suppose there are non-negative real numbers α, β and γ such that for any x, y in X and i, j $\in \aleph$, $d(T_i x, T_j y) \le \alpha\{d(x, T_i x) + d(y, T_j y)\} + \beta\{d(x, T_j y) + d(y, T_i x)\} + \gamma d(x, y)$, (4.1) where $2\alpha + 2\beta + \gamma < 1$ .Then $\{T_n\}_{n=1}^{\infty}$ has a unique fixed point.

In 1999, Sastry *et al.* established the following fixed point theorems for sequence of self maps.

THEOREM 4.1.[36] Let (X, d) be a bounded complete metric space and S and T be self maps of X such that ST = TS. Further, assume that S and T satisfy the following inequality: there exists

$k \in (0, 1)$ and $\varphi \in \Phi$ such that $\varphi((Sx, Ty)) \le k \max\{\varphi(d(x, y)), \varphi(d(x, Sx)), \varphi(d(y, Ty))\}$ (4.2)

for all x, y in X. Then one of S and T (and hence both) have a unique fixed point in X.

THEOREM 4.2.[36]: Let (X, d) be a bounded complete metric space. Suppose $\{T_i\}_{i=1}^{\infty}$ is a sequence of selfmaps of X such that $T_iT_j = T_jT_i$, for all i, j = 1, 2, 3… and satisfies the inequality :

there exists $k \in (0, 1)$ and $\varphi \in \Phi$ such that $\varphi(d(T_ix, T_jy)) \le k \max\{\varphi(d(x, y)), \varphi(d(x, T_ix)), \varphi(d(y, T_jy)\}$

for all x, y in X. Then, the sequence $\{T_i\}_{i=1}^{\infty}$ has a unique common fixed point in X.

In 2001, Babu and Ismail established the following fixed point theorems for single self maps.

THEOREM 4.4.[1]: Let (X, d) be a complete metric space, and T a selfmap of X. Assume that T satisfies the following inequality: there is a $k \in [0, 1)$ and $\psi \in \Psi$ such that

$\psi(d(Tx, Ty)) \le k \max\{\psi(d(x, y)), \psi(d(x, Tx)), \psi(d(y, Ty)), [\psi(d(x, Ty)) + \psi(d(y, Tx))]/2\}$, for all $x, y \in X$. For any $x_0 \in X$, define $x_n = T^n x_0$, n = 1, 2, … Then $\{x_n\}_{n=1}^{\infty}$ is Cauchy, $\lim_{n\to\infty} x_n$ exists, say z and z is the unique fixed point of T in X.

The Theorem 4.4 has been extended to the sequence of selfmap as follows.

THEOREM 4.5.[1]: Let (X, d) be a complete metric space, and $\{T_n\}_{n=1}^{\infty}$ be a sequence of selfmap of X. Suppose there is a $\psi \in \Psi$ satisfying the following inequality: there exists $k \in [0, 1)$ such that

$\psi(d(T_1x, T_jy)) \le k \max\{\psi(d(x, y)), \psi(d(x, T_1x)), \psi(d(y, T_jy)), [\psi(d(x, T_jy)) + \psi(d(y, T_1x))]/2\}$, for

all $x, y \in X$ and for all $j \in \aleph$. Then, the mappings $\{T_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

In 2002, K. Jha and R.P. Pant established the following fixed point theorems for sequence of self maps using reciprocally continuity.

THEOREM 4.6[14]: Let $\{A_i\}$, i = 1, 2, 3, …, S and T are self mappings on a complete metric space (X, d) for some i > 1 and $\Re$ is a control function as in (1.2) satisfying: (i) $A_1X \subset TX$, $A_2X \subset SX$, (ii) $\psi(d(A_1x, A_2y)) \le h\, M_{\psi_1}(x, y)$, $0 \le h < 1$, and (iii) $\psi(d(A_1x, A_iy)) < M_{\psi_1}(x, y)$, whenever $M_{\psi_1}(x, y) > 0$. Suppose that (A_1, S) and (A_2, T) be ψ-compatible pair of reciprocally continuous mappings. Then, all the A_i, S and T have a unique common fixed point. In 2005, Choudhari and Dutta established the following fixed point theorems for fuzzy mappings in complete metric space using generalized altering distance function.

THEOREM 4.7.[4]: Let (X, d) be a complete metric linear space and S, T : X $\rightarrow$ W(X) be two fuzzy mappings such that the following holds: For all $x, y \in X$,

$$\phi_1(d(Sx, Ty)) \leq \psi_1(d(x, y), D_1(x, Sx), D_1(y, Ty)) - \psi_2(d(x, y), D_1(x, Sx), D_1(y, Ty)),$$

where, ψ_1 and ψ_2 are generalized altering distance functions and $\phi_1(x) = \psi_1(x, x, x)$. (4.3)
Then S and T have a unique common fixed point.

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