TRAFFIC FLOW MODELS

Athma Ram Adhikari [1]

Abstract

Traffic flow is the study of the movement of individual drivers and vehicles between two points and the interactions they make with one another.Traffic flow models can be used to simulate traffic, for instance to evaluate ex-ante the use of a new part of the infrastructure. Models can be categorized based on, firstly, representation of the traffic flow in terms of flows (macroscopic), groups of drivers (macroscopic) or individual drivers (microscopic) and, secondly, underlying behavioral theory, which can be based on characteristics of the flow (macroscopic) or individual drivers (microscopic behavior).

Microscopic traffic flow variables focus on individual drivers. Macroscopic traffic flow variables reflect the average state of the traffic flow. A macroscopic traffic flow model is a mathematical traffic model that formulates the relationships among traffic flow characteristics like density, flow, mean speed of a traffic stream etc. such models are conventionally arrived at by integrating microscopic traffic flow models and converting the single-entity level characteristics to comparable system level characteristics.

Key Words: *Traffic, Traffic flow models, Macroscopic model, Microscopic model, Traffic speed, Traffic density, Traffic flow rate, Extreme point, Maximum flow rate, Linear car following model.*

Introduction

We like to drive our own vehicles on a traffic arteries. Sometimes we can drive in maximum velocity and sometimes this velocity approaches to zero. To analyze the traffic flow in arteries, we deal with two types of modeling.

They are:

i) Macroscopic modeling ii) Microscopic modeling

i) Macroscopic Modeling

The macroscopic modeling of traffic assumes a sufficiently large number of cars in a lane or on a road such that each stream of autos can be treated as we would treat fluid flowing in a tube or stream.

Macroscopic models are expressed in terms of three variables for a whole line of traffic:

Variables

1 **Rate of flow:** The number of cars passing a fixed point per unit time, called the rate of flow, denoted by $q(x, t)$.

1 Associate Professor, Tikapur Multiple Campus, Tikapur, Kailali
athmaram286@gmail.com

2 **Speed of traffic flow:** The distance covered per unit time by a car, called speed of traffic flow, denoted by v (x , t).

3 **Traffic density:** The number of cars per unit length of road, called traffic density, denoted by ρ(x, t).

Conservation of Cars

The conservation principle states that the change in the number of cars within that stretch of road results from the flow of traffic into and out of that road interval, and from the generation or consumption of cars within the interval. In horrific mega-accidents that occur during severe fogs or major storms, we assume that the cars are neither generated nor consumed within that interval

Mathematical Formulation and notations

- A particular stretch or interval of road Δx with end-points $x = x$ and $x = x + \Delta x$.
- The number of cars within this road interval Δx is given by $\Delta N(x, t)$.
- The change in the number of cars within the interval Δx, $\Delta N(x, t)$, during a time interval Δt is, in limit, equal to the rate of traffic flow, $q(x , t)$:

$$q(x,t) = \lim_{\Delta t \to 0} \frac{\Delta N(X.t)}{\Delta t} \quad (1)$$

- $\Delta N(x,t)$ is the difference between the number of cars going in and out of that stretch of road at each end, $N(x,t)$ and $N(x + \Delta x,t)$, respectively:

$$\Delta N(x,t) = N(x,t) - N(x + \Delta x,t) \quad (2)$$

- If Δx denotes the length of road interval that is traveled during the time Δt, then the speed of traffic flow, $v(x,t)$, in the interval:

$$v(x,t) = \left(\frac{\Delta X}{\Delta t}\right) \quad (3)$$

- Equation (1) can be written as

$$q(x,t) = \lim_{\Delta t \to 0} \frac{\Delta N(X,t)}{\Delta X}\left(\frac{\Delta X}{\Delta t}\right) \quad (4)$$

- From equations (2), (3) and (4),

$$q(X,t) = \left(\lim_{\Delta x \to 0} \frac{N(X,t) - N(X+\Delta X,t)}{\Delta X}\right) V(X,t) \quad (5)$$

- The number of cars per unit length of road, which we define as the traffic density:

$$q(x,t) = \lim_{\Delta X \to 0} \frac{N(X,t) - N(\Delta + X,t)}{\Delta X} \quad (6)$$

- Combining equation (5) and (6), we get,

$$q(X, t) = \rho(X, t)V(X, t) \quad (7)$$

- The number of cars, ΔN, passing a point during a time interval, Δt, is simply the product of the flow rate, q, and the time interval:

$$\Delta N = q\Delta t \quad (8)$$

- During the same small interval of time, a car moving with a speed v will cover a distance, $\Delta x = v\Delta t$.

- The number of vehicles passing through that distance is found from another simple product; density times distance:

  $\Delta N = \rho\Delta x$ (9)
- From equation (8) and (9), we get,

  $q\Delta t = \rho\Delta x$ (10)

  Since there is connection between traffic density and vehicle velocity, we can write,
  $v = v(\rho)$ (11)
  (Which means that the more vehicles are on the road, the slower their velocity will be.)
  Also, there is a relationship between the traffic flow rate q and the density ρ, so we write,

  $q(\rho) = \rho v(\rho)$ (12)

Relation between traffic speed and traffic density

Since there is a connection between traffic density and vehicle velocity, we can write,
$v = v\ (\rho)$

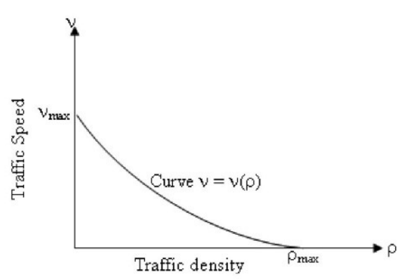
Also, a driver will drive fastest, v_{max}, when the density is at smallest value, (i.e. $\rho \to 0$).

Traffic grinds to a halt, $v = 0$, at jam density (ρ_{jam}).
Thus we summarize mathematically as follows:

$v(\rho = 0) = v_{max}$, (13a)

$\frac{dV}{d\rho} \leq 0$, (13b)

$v(\rho = \rho_{jam}) = 0$. (13c)



In this diagram, slope is always non-positive, $\frac{dV}{d\rho} < 0$, which results from our experience that speed drops off as traffic density increases.

Relation between traffic flow rate and traffic density

According to the traffic engineer, the most relevant variable is the capacity that the road system must accommodate, as reflected in traffic flow rate, q(X ,t).

We take the speed to be homogeneous, (i.e. it does not depend on the road coordinate, x, or on time, t).
So, we can write,
V = V (ρ),
Which means that traffic flow ultimately depends only on the density, ρ.
A driver's fastest speed, v_{max}, occurs when the density is at its smallest, ρ = 0.
$\therefore q(\rho = 0) = 0$, [∵from eq^n (12)
Which tells that flow rate is zero.
Similarly, traffic slows to a halt at its maximum density,

$V(\rho_{jam}) = 0$
$\therefore q(\rho_{jam}) = \rho_{jam} V(\rho_{jam}) = 0$,[∵from eq^n (13)
Which again tells that flow rate is zero.
The traffic flow rate must be positive for all values of the density ($0 < \rho < \rho_{jam}$), and must attain its maximum value q_{max} somewhere in that interval.
Now, diff. eq^n(12) w.r.t. ρ, we get

$$\frac{dq}{d\rho} = v(\rho) + \rho\frac{dV}{d\rho} \qquad (14)$$

Let us consider a linear speed-density relationship:

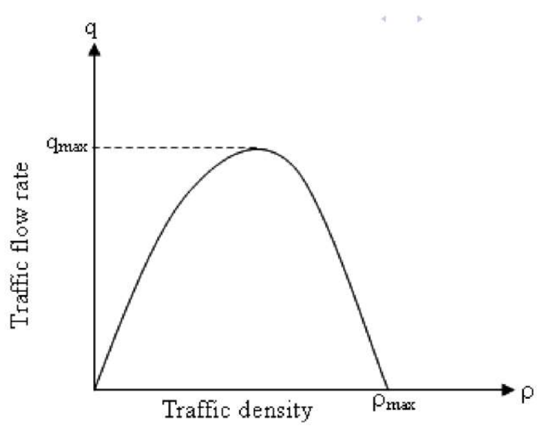
$$v(\rho) = vmax\left(1 - \frac{\rho}{\rho jam}\right) \qquad (15)$$

From eq^n(12) and eq^n(15), we have,

$$q(\rho) = Vmax\left(\rho - \frac{\rho^2}{\rho jam}\right) \qquad (16)$$

Which is parabolic as shown in the figure below.

Fundamental diagram of traffic flow



In this diagram, the slope is positive until the maximum flow rate, q_{max} , is reached, and negative thereafter.

Extreme point and maximum flow rate

For stationary points:

$$\frac{dq(\rho)}{d\rho} = V_{max}\left(1 - \frac{2\rho}{\rho_{jam}}\right) = 0 \qquad (17)$$

Also,

$$\frac{d^2q(\rho)}{d\rho^2} = -2(gives\ maximum)$$

Eqn (17) gives: $\rho = \frac{\rho_{jam}}{2}$

Thus, the maximum flow rate occurs at the mid-point of the fundamental diagram, and the value is

$$q_{max} = \frac{1}{4}\rho_{jam}V_{max}$$

ii) Microscopic model

Introduction

- Microscopic models look at individual cars.
- Microscopic models develop the traffic speed - traffic density relations.
- Microscopic models are the psychological models in order to model human behavior, because

i) The driver will perceive a variety of stimuli, including the distance between vehicles, their relative speed, and their perceived relative acceleration.

ii) The driver's response will depend on the responder's sensitivity to the given stimuli, as well as on the speed with which the response is undertaken.

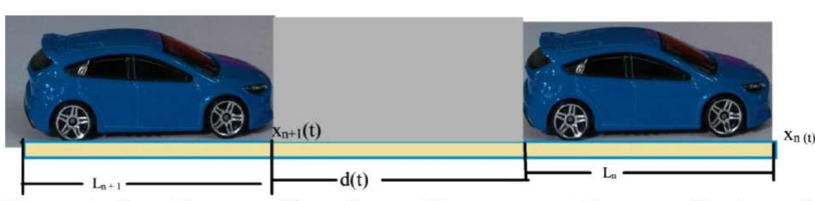
Linear car - following model

For this model, following assumptions are made: Assumptions

1 Each car has same length, L, and is separated from its neighbors by a common distance d(t).

2 Each car is identified by a discrete coordinate that varies in time, so that the location of n^{th} car is given by $x_n(t)$.

3 Not permitted overtaking.



Discrete functions, $x_n(t)$ and $x_{n+1}(t)$, represent the coordinates of the leader and follower cars.

Mathematical Formulation

- Basic equation of car - following model is the psychological one: response = sensitivity.stimulus (18)
- The response will generally be modeled as the acceleration of the (n+1)st follower car, $\ddot{x}_{n+1}(t)$, as it moves behind the nth car.

- The stimulus will be modeled in terms of the coordinate of the follower car relative to the leader car, which can be written in terms of traffic density, ρ.
- The acceleration is integrated to determine the speed of that car as a function of the traffic density.
- In simple car-following model, driver of the follower car responds to the speed of the leader car relative to follower car:

$$\frac{d^2X_{n+1}(t)}{dt^2} = -K_p\left(\frac{dX_{n+1}(t)}{dt} - \frac{dX_n(t)}{dt}\right)$$

Where the coefficient K_ρ is a sensitivity parameter that has dimensions of per unit time.

- The time it takes the following driver to respond to events by building in a reaction time that slows the follower's acceleration by the delay time T. So

$$\frac{d^2X_{n+1}(t+T)}{dt^2} = -K_\rho\left(\frac{dX_{n+1}(t)}{dt} - \frac{dX_n(t)}{dt}\right) \qquad (20)$$

Integrating (20), we get,

$$\frac{dX_{n+1}(t+T)}{dt} = -K_p\left(X_{n+1}(t) - X_n(t)\right) + C_{n+1}, \qquad (21)$$

Where C_{n+1} is the arbitrary constant.

- Eq^n (21) clearly relates the speed of the follower car to the distance maintained between the follower and leader cars.

- Distance between any two cars is given by d(t):
  $d(t) = x_n(t) - L - x_{n+1}(t)$. (22)
- The number of cars, N_R, found in a stretch of road of length, L_R, is
  $$N_R = \frac{L_R}{L+d(t)} \qquad (23)$$
- The density of the cars on the road is
  $$\rho = \frac{L_R}{N_R} = \frac{1}{L+d(t)} = \frac{1}{X_n(t)-X_{n+1}(t)} \qquad (24)$$
- In eq^n (24), we have a relationship between the (macroscopic) traffic density, ρ, and the (microscopic) coordinates of the leader and follower cars.
- If the traffic flow is in steady state or all cars are traveling at the same speed, then
  $$\frac{dX_{n+1}(t+T)}{dt} = \frac{dX_{n+1}(t)}{dt} \equiv V \qquad (25)$$

- Equation (25) shows a relationship between the (macroscopic) speed v and the (microscopic) speeds of any of the follower cars.
- From (21), (24) and (25), we get,
  $$V = \frac{K_P}{\rho} + C \qquad (26)$$

Where C is the arbitrary constant C_{n+1} in (21), which C can be determined from (13c) that is, the speed is zero when the density is at maximum or jam value.

- Hence we get,

$$v = K_p\left(\frac{1}{\rho} - \frac{1}{\rho_{jam}}\right) \qquad (27)$$

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