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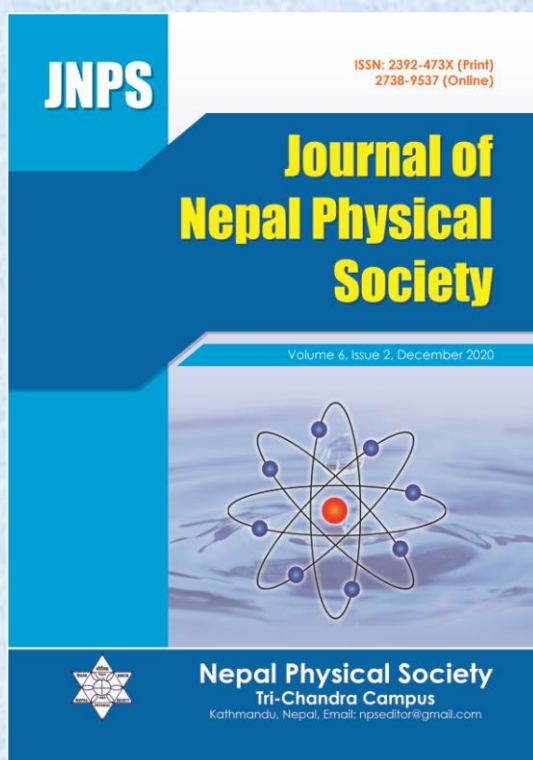
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Numerical Solution of Parabolic Partial Differential Equation by Using Finite Difference Method

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Abstract

In the real world, many physical problems like heat equation, wave equation, Laplace equation and Poisson equation are modeled by partial differential equations (PDEs). A PDE of the form $u_t = \alpha u_{xx}$, ($\alpha > 0$) where x and t are independent variables and u is a dependent variable; is a one-dimensional heat equation. This is an example of a prototypical parabolic equation. The heat equation has analytic solution in regular shape domain. If the domain has irregular shape, computing analytic solution of such equations is difficult. In this case, we can use numerical methods to compute the solution of such PDEs. Finite difference method is one of the numerical methods that is used to compute the solutions of PDEs by discretizing the given domain into finite number of regions. Here, we derived the Forward Time Central Space Scheme (FTCSS) for this heat equation. We also computed its numerical solution by using FTCCS. We compared the analytic solution and numerical solution for different homogeneous materials (for different values of diffusivity α). There is instantaneous heat transfer and heat loss for the materials with higher diffusivity (α) as compared to the materials of lower diffusivity. Finally, we compared simulation results of different non-homogeneous materials.

Keywords: Finite Difference Methods; Forward Time Central Space Scheme; Heat conduction in non-homogeneous material; Numerical solution; Partial Differential Equations; Thermal diffusivity.

1. INTRODUCTION

Heat is an important process of energy transfer in the case of temperature difference between two different points. In the one hand, the term 'heat' is used to describe the energy transferred through the heating process. On the other hand, temperature is a physical property of substance which describes the coldness or hotness of an environment or object [1, 21]. Heat transfer is a thermal engineering which concerns the generation, use, conversion and exchange of thermal energy and/or heat between physical systems. Heat transfer is generally classified into various modes, like heat conduction, convection, thermal radiation, and transfer of energy by phase change [1, 17]. The conduction mechanisms of heat transport occur either because of an exchange of energy from one molecule to another (without the actual motion of the

molecules), or because of the motion of the free electrons if they are present. So, these modes of heat transport depend upon the material properties like diffusivity of the medium [1, 17].

Differential equation is an equation involving derivative of an unknown function of one or more variables. Partial differential equations (PDEs) are important in many branches of science and engineering because there is always more than one independent variable involved in real life physical problems. There are very few PDEs that we can solve, mostly linear equations and some nonlinear equations [1, 2, 10, 12]. In Mathematics and Physics, the heat equation or diffusion equation is a PDE which describes the distribution of heat evolution over time in a solid medium [2, 10, 12, 23]. The heat equation is an important PDE which describes the variation in temperature (or

distribution of heat) in a given region over time [25]. The heat equation is very important in diverse scientific fields [2, 8, 10, 11, 12].

The diffusion equation is the more general version of the heat equation which arises in connection with the study of chemical diffusion and other related processes. The geothermal gases of constant heat or mass flux and the constant wall temperature or concentration are electrically conducting [2, 8, 10, 11, 12]. According to the study of Chamkha and Khaled [5], the effect of magnetic field on the coupled heat and mass transfer by mixed convection in a linearly stratified stagnation flow in the presence of an internal heat generation or absorption. The modeling and exact analytic solutions for hydromagnetic oscillatory rotating flows of an incompressible Burgers fluid bounded by a plate [8]. The heat equation was first developed by Jean Baptise Joseph Fourier (1768-1830) and presented as a manuscript to the Institute de France in 1807 AD and published in his monograph, Analytic Theory of Heat in 1822 AD [14]. In the 1600s, scientists appear to have thought correctly that heat is related with the motion of microscopic constituents of the matter. But it was believed, in the 1700s, that heat was a separate fluid-like substance [26].

If $u = u(x, t)$, then the partial differential equation of the form [2]

$$u_t(x, t) = \alpha u_{xx}(x, t), \quad \alpha > 0 \dots (1.1)$$

BCs: $u(0, t) = T_1, u(1, t) = T_2; t > 0$

IC: $u(x, 0) = f(x); 0 \leq x \leq L$

where T_1, T_2 are boundary temperatures, $f(x)$ is initial amount of heat given at the length x of the rod and α is thermal diffusivity, is called one dimensional heat equation or one-dimensional diffusion equation [4].

Comparing the above equation (1.1) with the second order linear PDE

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G \dots(1.2)$$

We get, $A = \alpha$ and $B = C = 0$.

$$\therefore B^2 - 4AC = 0$$

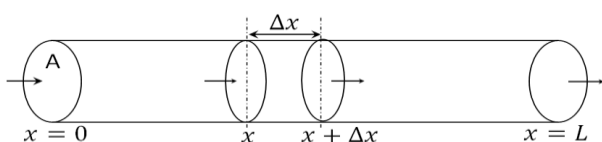


Fig. 1: Heat flow in a rod.

So, it is an example of a prototypical parabolic partial differential equation [1]. The Heat equation has analytic solution in regular shape domain whereas if the domain has irregular shape, computing analytic solution of such equations is very difficult [13]. To overcome such difficulty, we use numerical methods to compute the solution of the modeled partial differential equations. Finite difference method is one of the numerical methods that is applied to compute the solutions of partial differential equations by discretizing the domain into finite number of regions. The solutions are computed at the grid points of the domain [13].

Heat equation (Diffusion equation) is widely used in particle diffusion, Brownian motion, Schrödinger equation for a free particle and thermal diffusivity in polymers. It is also used to (i) maintain the temperature on the outer surface of rockets, which prevents the rocket from fire (ii) construction of railway tracks and bridges, which prevents the bending of tracks and bridges from expansion (iii) calculate the diffusion rate of diseases through air. Moreover, it is used in refrigerators, image analysis, cancer model and spatial ecological model [15].

The approach of our work is mainly focused on the numerical solution of one dimensional parabolic partial differential equation. The second section comprises formulation of numerical methods. The third section depicts the simulation results and comparison of solutions of materials of different diffusivity. The fourth section culminates the conclusion and finding of the work.

2. NUMERICAL METHODS

Various approximate methods have been investigated to solve the time dependent partial differential equations. In numerical analysis the analysis of these methods in terms of convergence, stability and order of accuracy is a main objective [21]. One way to solve time-dependent partial differential equations numerically is to discretize in space but leave time variable [9, 11, 16, 18, 21]. Here, we consider the Forward Time Central Space Scheme (FTCSS) and employed it to find the numerical solution.

The forward difference in time is given by [20],

$$u(x, t + \Delta t) = u(x, t) + \Delta t \frac{\partial u(x, t)}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u(x, t)}{\partial t^2} + \dots$$

After re-arrangement and dividing by Δt , we get

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{\partial u(x, t)}{\partial t} + O(\Delta t) \quad \dots (2.1)$$

Also, central difference in space is given by [20],

$$u(x + \Delta x, t) = u(x, t) + \Delta x \frac{\partial u(x, t)}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u(x, t)}{\partial x^3} + \dots$$

$$u(x - \Delta x, t) = u(x, t) - \Delta x \frac{\partial u(x, t)}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 u(x, t)}{\partial x^3} + \dots$$

Adding and re-arranging, we get,

$$\frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2} = \frac{\partial^2 u(x, t)}{\partial x^2} + O(\Delta x)^2 \quad \dots (2.2)$$

Substituting the value of (2.1) and (2.2) in equation (1.1), we get

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \alpha \left(\frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2} \right) + O(\Delta t, (\Delta x)^2)$$

Re-arranging we obtain,

$$u(x, t + \Delta t) \approx u(x, t) + \alpha \left(\frac{\Delta t}{(\Delta x)^2} \right) \{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)\} \quad \dots (2.3)$$

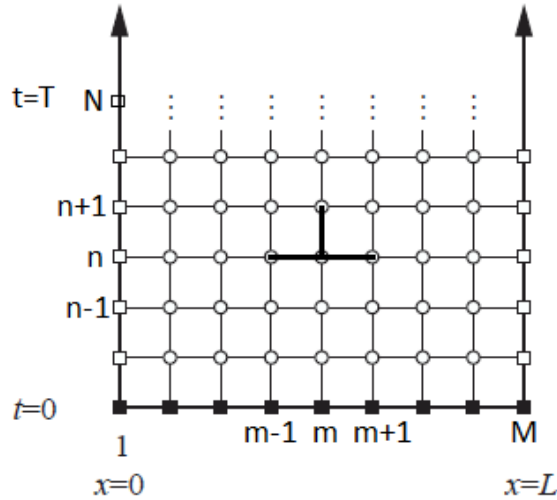


Fig. 2: Grid points of space-time plane [17].

We subdivide the spatial interval $[0, L]$ into $M + 1$ equally spaced sample points $x_m = m \cdot \Delta x = m \cdot h$. The time interval $[0, T]$ is subdivided into $N + 1$ equal time levels $t_n = n \cdot \Delta t = n \cdot k$ [3, 20]. At each of these space-time points we introduce approximations $u(x_m, t_n) \approx v_m^n$.

Now, equation (2.3) reduces to,

$$v_m^{n+1} = v_m^n + c (v_{m+1}^n - 2v_m^n + v_{m-1}^n) \quad \dots (2.4)$$

where, $c = \alpha \left(\frac{\Delta t}{(\Delta x)^2} \right) = \alpha \frac{k}{h^2}$

which is our required Forward Time Central Space scheme for (1.1) [18, 19, 20].

The numerical solution of the heat equation (1.1) can be found by using the initial and boundary conditions to the scheme (2.4). The Forward Time Central Space scheme (2.4) is consistent with the order of accuracy (1, 2) and is stable if and only if

$c \leq \frac{1}{2}$ [18, 20]. Thus, in order to use FTCS, we must have the value of $c = \alpha \frac{k}{h^2}$, which is defined above must be less than or equal to 0.5. We can maintain this condition by resizing the lengths of time and space intervals. To find the more accurate approximation we have to increase the number of space and time partitions.

3. SIMULATION RESULTS AND DISCUSSION

3.1 Comparison Between Analytic and Numerical Solution

Let us consider an example of heat equation as follows: [7]

$$u_t = 0.05 u_{xx}; \quad 0 \leq x \leq 1, \quad t \geq 0 \quad \dots (3.1)$$

BCs: $u(0, t) = u(1, t) = 0; \quad t > 0$ and

IC: $u(x, 0) = \sin \pi x; \quad 0 \leq x \leq 1$

3.1.1. Analytic Solution

By using the separation of variables method, the solution of 1D heat equation (3.1) is given by [7],

$$u(x, t) = (A \cos \lambda x + B \sin \lambda x) e^{-0.05 \lambda^2 t}$$

where, the constants A, B and separation constant $-\lambda^2$ are given by initial and boundary conditions [2, 11, 13].

Now, using the boundary conditions and superposition principle we get,

$$v_0^0 = 0, \quad v_1^0 = 0.5878, \quad v_2^0 = 0.9511, \quad v_3^0 = 0.9511, \quad v_4^0 = 0.5878, \quad v_5^0 = 0$$

For $m = 1$ and $n = 0$, we get

$$v_1^1 = v_1^0 + 0.25 (v_2^0 - 2 \times v_1^0 + v_0^0)$$

$$\therefore v_1^1 = 0.5878 + 0.25(0.9511 - 2 \times 0.5878 + 0) = 0.5317$$

Similarly, we have

$v_0^0 = 0$	$v_1^0 = 0.5878$	$v_2^0 = 0.9511$	$v_3^0 = 0.9511$	$v_4^0 = 0.5878$	$v_5^0 = 0$
$v_0^1 = 0$	$v_1^1 = 0.5317$	$v_2^1 = 0.8602$	$v_3^1 = 0.8602$	$v_4^1 = 0.5317$	$v_5^1 = 0$
$v_0^2 = 0$	$v_1^2 = 0.4809$	$v_2^2 = 0.7781$	$v_3^2 = 0.7781$	$v_4^2 = 0.4809$	$v_5^2 = 0$
$v_0^3 = 0$	$v_1^3 = 0.4350$	$v_2^3 = 0.7038$	$v_3^3 = 0.7038$	$v_4^3 = 0.4350$	$v_5^3 = 0$
$v_0^4 = 0$	$v_1^4 = 0.3934$	$v_2^4 = 0.6366$	$v_3^4 = 0.6366$	$v_4^4 = 0.3934$	$v_5^4 = 0$
$v_0^5 = 0$	$v_1^5 = 0.3559$	$v_2^5 = 0.5758$	$v_3^5 = 0.5758$	$v_4^5 = 0.3559$	$v_5^5 = 0$

Now, we find the error at $x = 0.8 m$ and $t = 0.8 hr$. From analytic solution we get $u_{exact} = u(0.8, 0.8) = 0.3961$.

Form FTCS approximate solution at $(0.8, 0.8)$
 $u_{approx} = v_4^4 = 0.3934$.

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-0.05(n\pi)^2 t}$$

Again, after using the initial conditions [6], we get

$$u(x, t) = \sin(\pi x) e^{-0.05\pi^2 t} \quad \dots (3.2)$$

Now, we are interested to find the temperature distribution of rod at distance $x = 0.8 m$ from the initial point of rod at time $t = 0.8 hr$. That is, we are interested to find the value of $u(0.8, 0.8)$.

From above solution (3.2), we get

$$u(0.8, 0.8) = 0.3961$$

3.1.2. Numerical Solution by Using FDM

The FTCS scheme of the above heat equation (3.1) is [20]

$$v_m^{n+1} = v_m^n + c(v_{m+1}^n - 2v_m^n + v_{m-1}^n)$$

with $v_0^n = v_M^n = 0, \quad v_m^0 = \sin(\pi m h)$ and $c = \frac{0.05 k}{h^2}$. Let the length of time and space intervals be $k = 0.2$ and $h = 0.2$ respectively. Then,

$$c = \frac{0.05 \times 0.2}{0.2^2} = 0.25$$

We know that, the FTCS scheme is stable iff $c \leq 0.5$, so our FTCS is stable for the above problem [20]. We have $v_m^0 = \sin(\pi m h)$, so

Thus, Error = $|u_{exact} - u_{approx}| = 0.0027$ and
 $\%Error = \frac{0.0027}{0.3961} \times 100\% = 0.68\%$

The 3D-plots of analytic and numerical solutions of equation (3.1) generated by computational software is shown in the Figure 3 below [22].

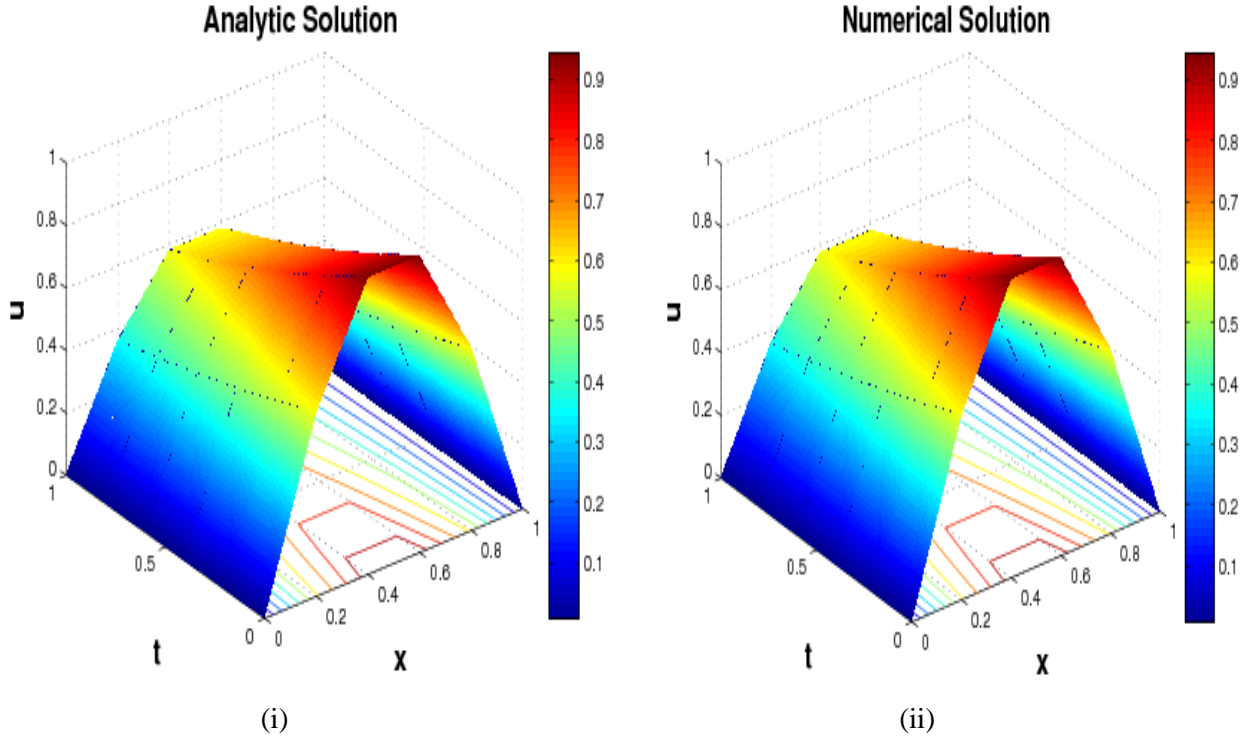


Fig. 3: (i) Analytic solution, (ii) Numerical solution of 1D heat equation (3.1).

3.2. Homogeneous Materials

In this section, we compare the analytic and numerical solution of heat equation in the case of rod of the three different homogeneous materials namely Nylon ($\alpha = 0.09 \text{ mm/s}^2$), Glass ($\alpha = 0.34 \text{ mm/s}^2$) and Quartz ($\alpha = 1.4 \text{ mm/s}^2$) [24].

In top panels (Fig. 4I), we clearly observed that it takes more time to distribute heat on the rod where as in middle panel (Fig. 4II) duration is lower moreover in the bottom of the panel (Fig. 4III) duration is substantially lower. Thus, we conclude that lower diffusivity means slower heat distribution in the rod and hence the duration of time taken to cool down the material is longer in

case of low diffusivity material as compared to the higher diffusivity materials. Hence, there is instantaneous heat transfer and heat loss for the materials with higher diffusivity (α).

Also, we observed that temperature of the substance is increasing faster in the middle panels of Fig. 4II as compared to the top panels of Fig. 4I and temperature of the substance is also increasing more quickly in the top panels of Fig. 4III as compared to middle panels of Fig. 4II. So, we conclude that in the substance having higher diffusivity the temperature of the substance quickly rises as compared to the substance having lower diffusivity. Hence, the temperature distribution of the rod $u(x,t)$ is directly proportional to the diffusivity of the materials.

Table 1: Comparison of analytic and numerical solution at $x = 0.8 \text{ mm}$ and $t = 2 \text{ sec}$.

Substances	Diffusivity (mm/s^2)	Analytic Solution	Numerical Solution	Error	% Error
1. Nylon	0.09	0.0995	0.0998	0.003	0.30%
2. Glass	0.34	7.1536×10^{-4}	7.1988×10^{-4}	4.52×10^{-6}	0.63%
3. Quartz	1.4	5.8551×10^{-13}	5.4551×10^{-13}	0.4×10^{-13}	6.83%

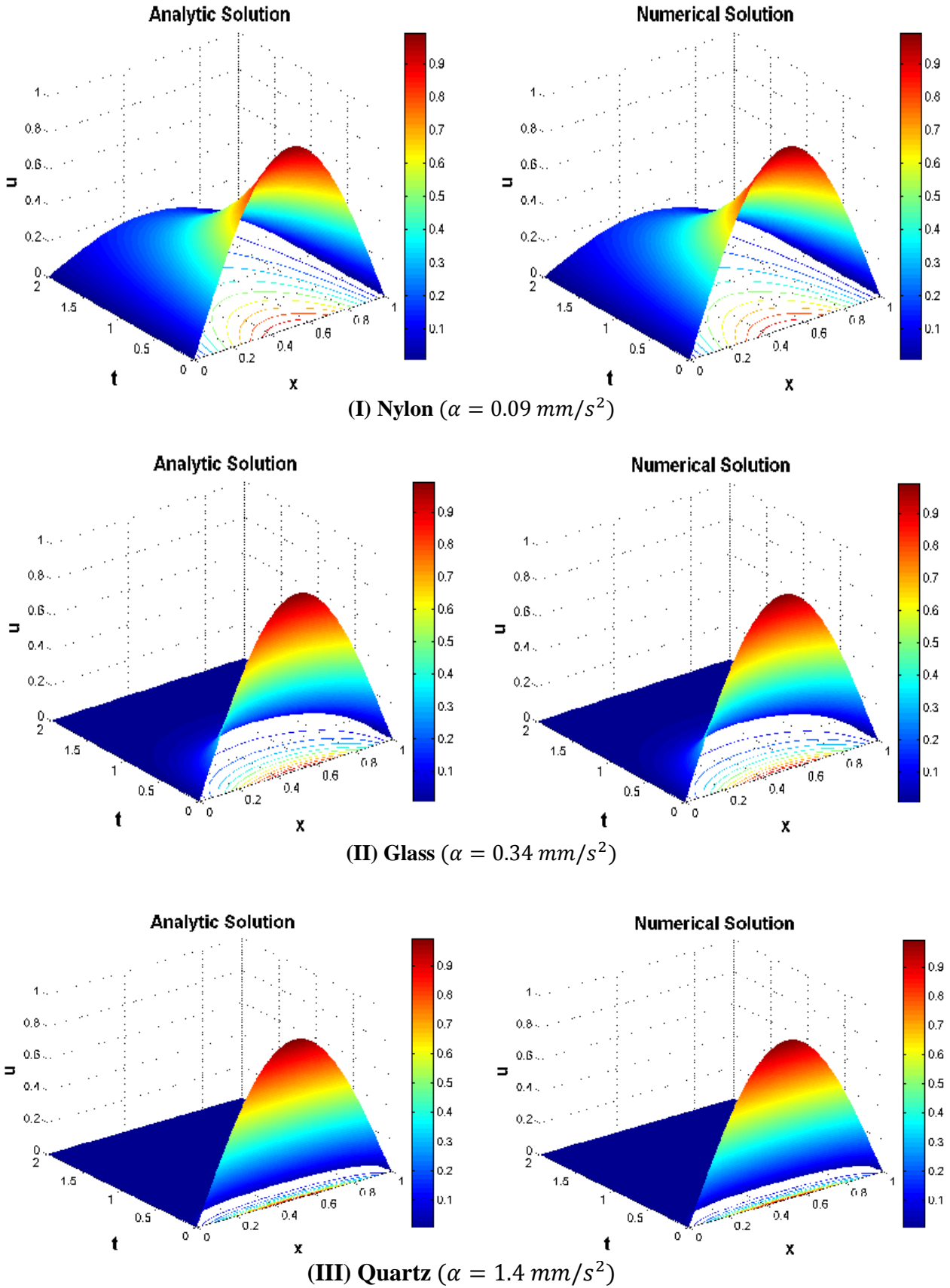


Fig. 4: Analytic and numerical solution (with $m = 20$ and $n = 3000$) of heat equation (1.1) with $T_1 = T_2 = 0$, $f(x) = T_0 = \sin \pi x$ for different values of α .

From table 1, percentage error in the case of substances Nylon, Glass and Quartz are 0.30%, 0.63% and 6.83% respectively.

So, we also conclude that materials having higher diffusivity has more error in numerical solution as compared to the exact solution. So numerical solution is much more appropriate in case of substance having lower diffusivity. We conclude that FTCS scheme gives the better approximation for the substances with smaller value of diffusivity as compared to the substances of higher diffusivity.

3.3. Non-Homogeneous Materials

In nature the materials found are generally non-homogeneous. Here, we consider heat flow in a one-dimensional rod composed of two different materials. We compare the simulation results in the case of heat flow from low diffusivity material to high diffusivity and high diffusivity material to low diffusivity whereas total length of a rod is kept constant ($L = 1m$) with homogeneous radius and composed of two substances of equal length (i.e., each substance having length $0.5 m$).

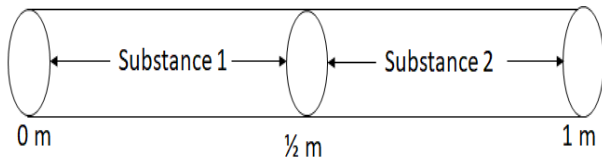


Fig. 5: Non-homogeneous rod composed of two different substances.

The corresponding PDE of this problem is

$$u_t = \begin{cases} \alpha_1 u_{xx} & \text{for } 0 \leq x \leq 0.5, \\ \alpha_2 u_{xx} & \text{for } 0.5 \leq x \leq 1 \end{cases} \dots (3.3)$$

where, α_1 and α_2 are diffusivity of substances 1 and 2 respectively. Also consider,

BCs: $u(0, t) = 0 = u(1, t)$ for $t > 0$

IC: $u(x, 0) = \sin \pi x$ for $0 \leq x \leq 1$

First, we consider substance 1 be Nylon and substance 2 be Quartz. After that we consider substance 1 be Quartz and substance 2 be Nylon. The diffusivity of Nylon is $0.09 mm/s^2$ and Quartz is $1.4 mm/s^2$. In Fig. 6a, we consider substance 1 be Nylon ($\alpha_1 = 0.09 mm/s^2$) and substance 2 be Quartz ($\alpha_2 = 1.4 mm/s^2$) [24]. So, heat flow from low diffusivity substance to high diffusivity substance. In this case, due to low diffusivity of left part of the material (substance 1) the heat flows forward slowly and hence temperature is slowly increased and the due to high diffusivity of right part of the material (substance 2) the heat flow is faster and hence temperature is quickly increased. But in Fig. 7b, we augmented substance 1 be Quartz ($\alpha_2 = 1.4 mm/s^2$) and substance 2 be Nylon ($\alpha_1 = 0.09 mm/s^2$) [24]. In this case heat flow from high diffusivity substance to low diffusivity substance.

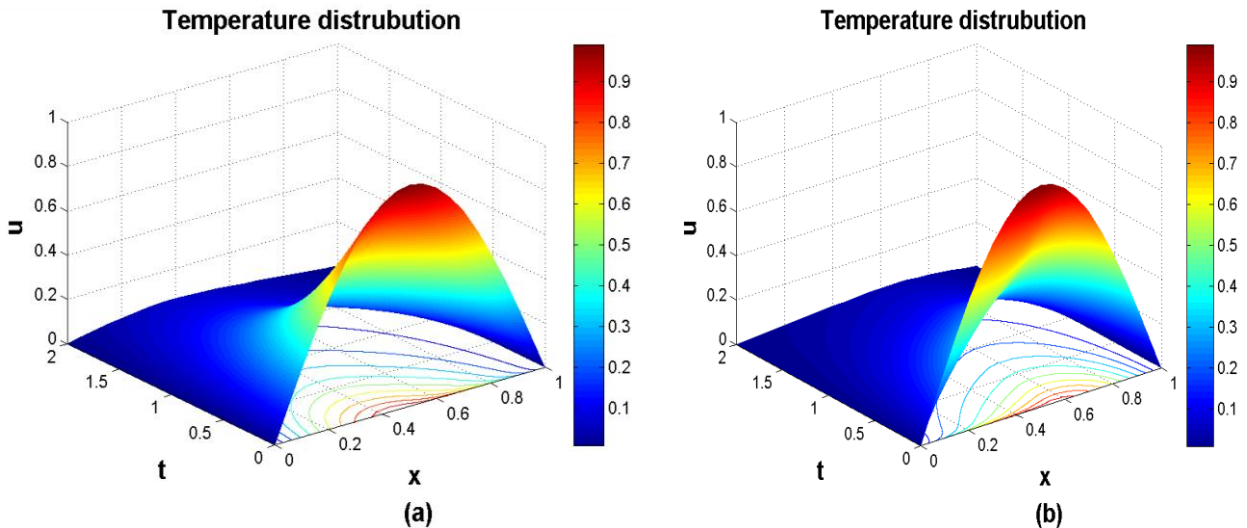


Fig. 6: Temperature distribution with (a) Nylon at left and Quartz at right (b) Quartz at left and Nylon at right [22].

Figure 7 is the plot of temperature distribution (u) versus x at different time levels $t = 0.5s, 1s, 1.5s$ and $2s$ with (a) Nylon at left and Quartz at right (b) Quartz at left and Nylon at right. In each time, maximum of temperature distribution (u) are around $x = 0.4m$ and $x = 0.65m$ in Fig. 7a and Fig. 7b respectively. This is because of quickly heat flow in left half part of Fig. 7a (low diffusivity materials in left part) and slowly heat flow in left half part of Fig. 7b (high diffusivity materials in left

part). In time $t = 2s$, maximum value of u higher in Fig. 7a [0.485] as compare to Fig. 7b [0.435] due to slowly distributed temperature in case left help part of Fig. 7a (low diffusivity materials in left part). We conclude that heat flow is quicker in the part consisting of material of higher diffusivity and heat flow is slower in the part consisting of material of lower diffusivity. This computational technique can be applicable and extended to the non-homogeneous materials found in nature.

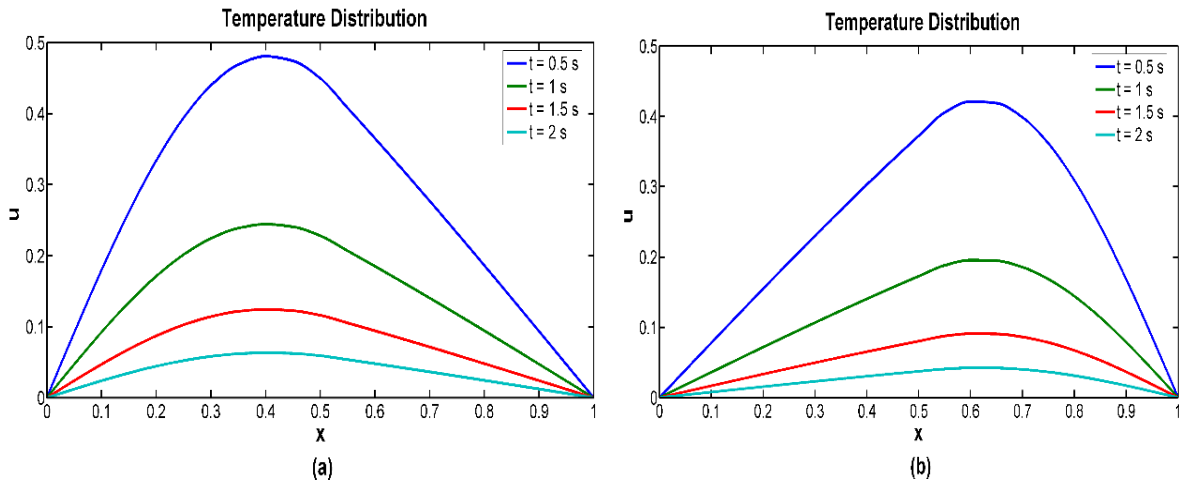


Fig. 7: Plot of temperature (u) versus x at different time levels $t = 0.5s, 1s, 1.5s$ and $2s$ with (a) Nylon at left and Quartz at right (b) Quartz at left and Nylon at right.

4. CONCLUSION

Here, at first one dimensional heat equation was introduced along with a bit of its history and its analytic solution was found by using separation of variable method. Then we derived Forward Time Central Space Scheme (FTCSS) for 1D heat equation, discussed their stability, consistency and numerical solution was found by using FTCS. And then by taking an example we compared the analytic and numerical solutions along with their corresponding plots. After this, we studied heat transfer on the homogeneous materials having different diffusivity and we conclude in the homogeneous substance having higher the diffusivity the temperature rises quickly as compared to the substance having lower diffusivity. There is instantaneous heat transfer and heat loss for the materials with higher diffusivity (α). Our numerical solution in case of homogeneous materials show that material having higher diffusivity have more error in numerical solution as compared to the exact solution. So, FTCS scheme gives the better approximation for the substances

with smaller value of diffusivity. In case of non-homogeneous materials, temperature quickly increases in part of the rod of the materials having higher diffusivity whereas temperature slowly increases in part of the rod of the materials having lower diffusivity. Our computational technique of can be applicable and extended to the non-homogeneous materials found in nature.

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