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Research Article

Special Cases on the Product of Two Confluent Hypergeometric Functions

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ABSTRACT

The hypergeometric function, represented as ${}_2F_1(a,b;c;z)$, is a specialized function that emerges in various fields including probability, combinatorics, engineering and mathematical physics. It serves as a generalization of the geometric series and fulfills a second-order linear differential equation. The product of hypergeometric functions frequently arises, particularly in relation to summation identities, integral transforms, or representation theory. These products can uncover profound symmetries and connections among special functions, playing a crucial role in addressing intricate challenges in both pure and applied mathematics. This study, building on the contributions of Poudel et al., seeks to elucidate specific cases of various product formulas assigning the different values of the parameters that involved in two confluent hypergeometric functions.

Keywords: Confluent hypergeometric function, exponential function, generalized hypergeometric function, hypergeometric function.

INTRODUCTION

John Walls extended the geometric series

$$1 + a + a^2 + a^3 + \ldots\ldots \qquad (1.1)$$

in the form

$$1 + a + a(a+b) + a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) + \ldots \qquad (1.2)$$

and presented his work in the book Arithmatica Infinitorium in 1655 (Rainville, 1971) which is now known as hypergeometric series with the n^{th} term

$$a(a+b)(a+2b)(a+3b)\,\ldots\ldots\ldots[a+(n-1)b] \qquad (1.3)$$

If b = 1, the expression (1.3) reduces to

$$a(a+1)(a+2)(a+3)\ldots\ldots\ldots\ldots\ldots[a+(n-1)] \qquad (1.4)$$

Then, the expression (1.4) can be denoted by

$$\begin{aligned}(a)_n &= a(a+1)(a+2)(a+3)\ldots\ldots\ldots\ldots[a+(n-1)] \\ &= \frac{\Gamma(a+n)}{\Gamma(a)},\ n \in Z+ \qquad (1.5) \\ &= \begin{cases} 1 & (n=0; a \in C\setminus\{0\}) \\ a(a+1)(a+2)\ldots\ldots[a+(n-1)] & (n \in N, a \in C) \end{cases}\end{aligned}$$

The symbol $(a)_n$ used in (1.5) to denote the n^{th} term of hypergeometric series is called the pochhmmer symbol.

According to Rainville (1971), Euler introduced the power series in the form

$$1+\frac{ab}{c}\times\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\times\frac{x^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\times\frac{x^3}{3!}\ldots\ldots. \qquad (1.6)$$

The expression (1.6) denoted by

$${}_2F_1(a, b; c; x) = {}_1F_1\begin{bmatrix} a, & b & \\ & c & ;x \end{bmatrix} = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}.$$

where *a, b, c* are the rational parameters, is called hypergeometric function and may undergo several transformation (Poudel, Harsh, & Pahari, 2023). The Gaussian hypergeometric function ${}_2F_1$ can be expressed in terms of gamma function as follows;

$${}_2F_1(a,\ b;\ c;\ 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (1.7)$$

The hypergeometric functions are the extension of the geometric series (Poudel, Harsh, Pahari, & Panthi, 2023). The generalized hypergeometric function ${}_pF_q$ having p number of numerator and q number of denominator parameters has been defined by Virchenko, Kalla, and Al-Zamel, (2001) as

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};\quad x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{x^n}{n!} \tag{1.8}$$

If p = q =1, the result of (1.8) reduces to

$$ {}_1F_1\left[\begin{matrix} a \\ b \end{matrix};x\right]=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{x^n}{n!}. \tag{1.9}$$

According to Kummer (1836), (1.9) is the confluent hypergeometric function. Kummer has also obtained the solutions of hypergeometric differential equations by using the theory of differential equations (Kummer, 1836; Poudel, Harsh, Pahari, & Panthi, 2023; Poudel, Pahari, Basnet, & Poudel, 2023). Further, he obtained the following results (Kim, Rakha, & Rathie, 2010; Rainville, 1971).

$$ e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix};\quad x\right]={}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix};\quad \frac{x^2}{16}\right] \tag{1.10}$$

In 1995, Rathie and Nagar established the following contiguous relation to Kummer's second Theorem.

$$ e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a \\ 2a-1 \end{matrix};\quad x\right]={}_0F_1\left[\begin{matrix} - \\ a-\frac{1}{2} \end{matrix};\quad \frac{x^2}{16}\right]+\frac{x}{2(2a-1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix};\quad \frac{x^2}{16}\right] \tag{1.11}$$

In 2023, Kim et al. established the following theorems by using the contiguous functions relations.

$$ e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix};\quad x\right]={}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix};\quad \frac{x^2}{16}\right]-\frac{x}{2(a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix};\quad \frac{x^2}{16}\right] + \frac{ax^2}{4(a+1)(2a+1)(2a+3)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix};\quad \frac{x^2}{16}\right] \tag{1.12}$$

Bailey (1928) has derived the product of two ${}_0F_1$ functions as

$$ {}_0F_1\left[\begin{matrix} - \\ a \end{matrix};\quad x\right]{}_0F_1\left[\begin{matrix} - \\ b \end{matrix};\quad x\right]={}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b), & \frac{1}{2}(a+b-1) \\ a, & b, & a+b-1 \end{matrix};\quad 4x\right] \tag{1.13}$$

Also, the exponential series in terms of hypergeometric function is

$$e^x = {}_0F_0\left[\begin{matrix} - \\ - \end{matrix} \quad ; x\right] \tag{1.14}$$

Several studies are done in the product of generalized hypergeometric functions. In 2024, Poudel et al. has developed five theorems with a corollary. Among them, theorem 2 is given below which is based on mainly the relations from (1.10) to (1.13) in the product form of two hypergeometric functions.

$$
\begin{aligned}
&{}_1F_1\left[\begin{matrix} a \\ 2a-1 \end{matrix} \; ; \; x\right] \times {}_1F_1\left[\begin{matrix} b \\ 2b+2 \end{matrix} \; ; \; x\right] \\
&= e^x \times \left\{ {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b), \; \frac{1}{2}(a+b-1) \\ a-\frac{1}{2}, \; b+\frac{1}{2}, \; a+b-1 \end{matrix} \; ; \; \frac{x^2}{4}\right] \right. \\
&\quad - \frac{x}{2} \times \left\{ \frac{1}{(b+1)} \, {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+1), \; \frac{1}{2}(a+b) \\ a-\frac{1}{2}, \; b+\frac{3}{2}, \; a+b \end{matrix} \; ; \; \frac{x^2}{4}\right] - \frac{1}{(2a-1)} \right. \\
&\quad \left. {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+1), \; \frac{1}{2}(a+b) \\ a+\frac{1}{2}, \; b+\frac{1}{2}, \; a+b \end{matrix} \; ; \; \frac{x^2}{4}\right] \right\} \\
&\quad + \frac{x^2}{4(b+1)} \times \left\{ \frac{b}{(2b+1)(2b+3)} \, {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+2), \; \frac{1}{2}(a+b+1) \\ a-\frac{1}{2}, \; b+\frac{5}{2}, \; a+b+1 \end{matrix} \; ; \; \frac{x^2}{4}\right] \right. \\
&\quad \left. \left. - \frac{1}{(2a-1)} \, {}_2F_3\left[\begin{matrix} \frac{1}{2}(a+b+2), \; \frac{1}{2}(a+b+1) \\ a+\frac{1}{2}, \; b+\frac{3}{2}, \; a+b+1 \end{matrix} \; ; \; \frac{x^2}{4}\right] \right\} \right\}
\end{aligned}
$$

$$+\frac{bx^3}{8(2a-1)(b+1)(2b+1)(2b+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(a+b+3), & \frac{1}{2}(a+b+2) & \\ a+\frac{1}{2}, & b+\frac{5}{2}, & a+b+2\end{matrix};\ \frac{x^2}{4}\right]\Bigg]\Bigg\} \tag{1.15}$$

RESULTS AND DISCUSSION

Following the precedent set by Tripathi (2025) for the first theorem of Poudel et al. (2024), we focus here on obtaining analogous formulation for their second theorem of the same authors. So assigning specific parameter values a = 1, 2, 3, 4 and b =1, 2, 3, 4 and x = 1 to equation (1.15), we derive a set of sixteen explicit results. These results express the second theorem in the elementary exponential form for the chosen parameters set which are as follows;

1. $${}_1F_1\left[\begin{matrix}1\\1\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}1\\4\end{matrix};1\right] = 3e(2e-5) \tag{2.1}$$

2. $${}_1F_1\left[\begin{matrix}1\\1\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}2\\6\end{matrix};1\right] = -20e(18e-49) \tag{2.2}$$

3. $${}_1F_1\left[\begin{matrix}1\\1\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}3\\8\end{matrix};1\right] = 210e(252e-685) \tag{2.3}$$

4. $${}_1F_1\left[\begin{matrix}1\\1\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}4\\10\end{matrix};1\right] = -324e(4580e-1234) \tag{2.4}$$

5. $${}_1F_1\left[\begin{matrix}2\\3\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}1\\4\end{matrix};1\right] = 6(2e-5) \tag{2.5}$$

6. $${}_1F_1\left[\begin{matrix}2\\3\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}2\\6\end{matrix};1\right] = -40(18e-49) \tag{2.6}$$

7. $${}_1F_1\left[\begin{matrix}2\\3\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}3\\8\end{matrix};1\right] = 420(252e-685) \tag{2.7}$$

8. $${}_1F_1\left[\begin{matrix}2\\3\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}4\\10\end{matrix};1\right] = -6048(4580e - 12341) \qquad (2.8)$$

9. $${}_1F_1\left[\begin{matrix}3\\5\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}1\\4\end{matrix};1\right] = 36(6e^2 - 31e + 40) \qquad (2.9)$$

10. $${}_1F_1\left[\begin{matrix}3\\5\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}2\\6\end{matrix};1\right] = -240(54e^2 - 291e + 392) \qquad (2.10)$$

11. $${}_1F_1\left[\begin{matrix}3\\5\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}3\\8\end{matrix};1\right] = 2520(756e^2 - 4071e + 5480) \qquad (2.11)$$

12. $${}_1F_1\left[\begin{matrix}3\\5\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}4\\10\end{matrix};1\right] = -36288(13740e^2 - 73663e + 98728) \qquad (2.12)$$

13. $${}_1F_1\left[\begin{matrix}4\\7\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}1\\4\end{matrix};1\right] = -360(64e^2 - 334e + 435) \qquad (2.13)$$

14. $${}_1F_1\left[\begin{matrix}4\\7\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}2\\6\end{matrix};1\right] = 2400(576e^2 - 3134e + 4263) \qquad (2.14)$$

15. $${}_1F_1\left[\begin{matrix}4\\7\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}3\\8\end{matrix};1\right] = -25200(8064e^2 - 43844e + 59595) \qquad (2.15)$$

16. $${}_1F_1\left[\begin{matrix}4\\7\end{matrix};1\right]\times{}_1F_1\left[\begin{matrix}4\\10\end{matrix};1\right] = 362880(146560e^2 - 793372e + 107366 \qquad (2.16)$$

The above results can also be verified by (1.14)

Proof of (2.1)

i) Finding the result of $\begin{bmatrix} 1 & \\ & ;1 \\ 1 & \end{bmatrix} = {}_1F_1(1;1;1)$

From (1.9), the confluent hypergeometric function in terms of the parameters a and b is ${}_1F_1(a;b;x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(b)_m} \cdot \frac{x^m}{m!}$ (2.17)

Also, from (1.5) $(a)_m = a(a+1)(a+2)(a+3)\text{............}(a+m-1)$

For ${}_1F_1(1;\ 1;\ 1)$, we have a = 1 and b = 1 and x = 1

So, from (2.17), $${}_1F_1(1;1;1) = \sum_{m=0}^{\infty} \frac{(1)_m}{(1)_m} \cdot \frac{1^m}{m!} = \sum_{k=0}^{\infty} \frac{1}{m!} = e \tag{2.18}$$

ii) Finding the result of $\begin{bmatrix} 1 & \\ & ;1 \\ 4 & \end{bmatrix} = {}_1F_1(1;4;1)$

Now, for ${}_1F_1(1;\ 4;\ 1)$, we have a = 1 and b = 4 and x = 1

So, from (2.17), ${}_1F_1(1;4;1) = \sum_{m=0}^{\infty} \frac{(1)_m}{(4)_m} \cdot \frac{1^m}{m!}$, where

$$\begin{aligned} &(1)_m = 1.2.3.....m = m!\, and \\ &(4)_m = 4.5.6.7.....(4+m-1) = \frac{(3+m)!}{3!} \end{aligned} \tag{2.19}$$

Then (2.19) reduces to

$$\begin{aligned} {}_1F_1(1;4;1) &= \sum_{k=0}^{\infty} \frac{m!}{\frac{(3+m)!}{3!}} \cdot \frac{1^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{6}{(3+m)!} \end{aligned} \tag{2.20}$$

In (2.20), let n = m + 3, then m = 0 and n = 3. The series becomes

$$ {}_1F_1(1;4;1) = \sum_{n=3}^{\infty} \frac{6}{n!} = 6(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{2} \frac{1}{n!}) \tag{2.21} $$

Then $\sum_{n=0}^{\infty} \frac{1}{n!} = e, and \sum_{n=0}^{2} \frac{1}{n!} = 1 + 1 + \frac{1}{2} = \frac{5}{2}$

Thus, from (2.21), ${}_1F_1(1;4;1) = 6(e - \frac{5}{2}) = 6e - 15 = 3(2e - 5)$

Hence, $\begin{bmatrix} 1 & ;1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & ;1 \\ 4 & \end{bmatrix} = 3e\left(2e - 5\right)$

By similar approach, the proofs of other relations can be performed.

CONCLUSION

This study establishes a set of sixteen distinct and novel results regarding the product of two confluent hypergeometric functions ${}_1F_1\begin{bmatrix} a & ; & x \\ 2a-1 & & \end{bmatrix}$ and ${}_1F_1\begin{bmatrix} b & ; & x \\ 2b+2 & & \end{bmatrix}$ by utilizing summation identities, integral transforms, and other known product formulae in the form of elementary exponential function by taking the particular values of the parameters a = 1, 2, 3, 4, b = 1, 2, 3, 4 and x = 1 and verified the result of (1.15) and cross-checked numerically with Wolfram Mathematica ensuring conceptual consistency and correctness. Taking other certain values of the parameters, further results can be obtained. The findings will contribute to a deeper understanding of hypergeometric function theory and enrich Engineering Science, Statistical Distribution Theory, Computational Mathematics and Mathematical Physics.

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