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Research Article

A LIMIT LAW FOR SUMS OF STOPPING TIME INDEXED SIGNUM FUNCTIONS

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ABSTRACT

A variety of limit laws describing the asymptotic behavior of functions have been developed in mathematics and statistics, among which the law of the iterated logarithm is considered one of the most significant. Motivated by these foundational results, we establish a LIL for sums of signum functions indexed by stopping times and determine a precise upper bound for the associated limit law.

Keywords: Lacunary trigonometric series, Law of the iterated logarithm, Martingale, Signum function, Stopping time.

INTRODUCTION

In probability theory, a limit law, known as the LIL, provides a framework for understanding the precise long-term behavior of random variables sequences. It is closely connected to two classical limit laws, the central limit theorem and the law of large numbers, providing a more precise characterization of fluctuations in cases where the sample mean exhibits small variations but the potential for large deviations remains. The concept of the LIL was first proposed by A. Khintchine (Khintchine, 1924) in the study of the deviation of sums of Bernoulli random variables from their expected values. Specifically, Khintchine established the first form of the LIL to identify the precise rate of convergence in Borel's theorem. An important implication of Khintchine's result is Borel's theorem, which is an immediate consequence of it. Khintchine originally established the LIL for the specific case of Bernoulli random variables. A few years later, Kolmogorov (1929) extended Khintchine's findings to a broader class of functions of independent random variables. Since Kolmogorov's formulation, numerous analogues and extensions of the LIL have been developed in various branches of mathematics and statistics, including lacunary trigonometric series (Salem & Zygmund, 1950; Ghimire & Moore, 2014), Bloch functions (Przytycki, 1989), harmonic functions (Bańuelos et al., 1988; Bañuelos & Moore, 1991), martingales (Stout, 1970), (Ghimire, 2022), linear processes (Liu & Zhang, 2021), signum functions (Ghimire, 2024; 2025) among others. For a comprehensive overview of the LIL in diverse contexts, readers are referred to the survey paper by N. H. Bingham (Bingham, 1986). Motivated by the above developments, the main purpose of this article is to establish a LIL for tail sums of signum functions indexed by stopping times. In comparison to the classical and tail versions of the LIL for deterministic indices, our work incorporates random stopping times and estimates a precise upper bound for the associated limit law. The results obtained here extend earlier tail LIL results for signum functions and martingales to the setting of a stopping-time- indexed framework.

MATERIALS AND METHODS

Models and data

In this section, we discuss the mathematical framework, preliminary definitions and some known results that form the basis for the mathematical analysis carried out in this article. We recall several theorems and results that are essential tools for establishing our main results. These results are included only for completeness and to establish notation. We begin with the first LIL result of Khintchine (2024):

Theorem 1.1 Let $N_n(x)$ denote the count of digit 1's appearing in the first positions of the binary expansion of a number $x \in [0,1)$. Then for almost every x, we have

$$\limsup_{m\to\infty} \frac{|S_m(x)|}{\sqrt{2m \ln\ln m}} = 1$$

where $S_m(x) = \sum_{k=1}^{m}(2b_k(x) - 1)$ and b_k is the k^{th} digit in the binary expansion of $x \in [0,1)$.

This result was then extended by Kolmogorov (1929) to large class of independent random variables. His result is:

Theorem 1.2 Let $S_n = \sum_{i=1}^{n} X_i$ is a sequence of independent and identically distributed random variables. Assume that $|X_n|^2 \le \frac{\varepsilon_n n}{\log\log n}$ for some $\epsilon_n \to 0$. Then for a.e. ω, we have

$$\limsup_{m\to\infty} \frac{|S_n(\omega)|}{\sqrt{2n \ln\ln n}} = 1.$$

This form of the LIL characterizes the extent to which the sum of independent random variables can deviate from its mean (zero), with the magnitude of this deviation being approximately of the order $\sqrt{\log\log n}$ where n represents the variance of the partial sums. Because of the appearance of the iterated logarithm term, $\log\log n$ this theorem is referred to as the law of the iterated logarithm. For the background information on the previous findings on the LIL, we first discuss the analog of Kolmogorov's LIL in dependent structures and begin with the context of martingales. The term martingale originally referred to a betting strategy where a gambler doubles his bet after each loss, aiming to recover all losses with a single win. In a coin toss game, this strategy seems foolproof if the gambler has unlimited wealth, as a win will eventually occur. However, in reality, no one has an infinite amount of money, and the rapidly increasing bets can lead to bankruptcy. This idea is mathematically modeled by a type of stochastic process called martingales. In this case, a similar LIL was obtained by W. Stout (Stout, 1970), given by:

Theorem 2.1 If $\{g_n\}_{n=0}^{\infty}$ is a sequence of dyadic martingales on [0,1) then for almost every t on the set where $\{g_n\}$ is unbounded, we have

$$\limsup_{n\to\infty} \frac{|g_n(x)|}{S_n g(x)\sqrt{2\log\log S_n g(x)}} \le 1.$$

Thus, we observe that the classical LIL has been extended to settings with dependent structures, such as martingales, where the partial sums are no longer independent. This result characterizes the growth rate of martingale sequences and serves as a valuable foundation for investigating more refined versions of the LIL in cases where the underlying random variables exhibit dependence. We next discuss the setting of lacunary trigonometric series which represents the first analytical framework in which the LIL was established. This series is also considered as weakly dependent because of its behavior. We now define the series:

Definition 2.2 A $q-$ lacunary series is defined as $S_m(\theta) = \sum_{k=1}^{m}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ satisfying the lacunary condition $\frac{n_{k+1}}{n_k} > q > 1$.

Motivated by Kolmogorov's formulation of LIL, Salem and Zygmund (SZ) (1950) established a corresponding version for lacunary series. Their result is widely regarded as the first significant contribution to the analytic form of the LIL (Salem & Zygmund, 1950). In the case of integer values of n_k, we have

Theorem 2.3 If for $B_m^2 = \frac{1}{2}\sum_{k=1}^{m}(|a_k|^2 + |b_k|^2)$ and $M_m = \max_{1\le k\le m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$, and $\lim_{m\to\infty} B_m = \infty$ and S_m satisfying $M_m^2 \le K_m \frac{B_m^2}{\log\log(e^e + B_m^2)}$ for $K_m \downarrow 0$, then a.e. θ in unit circle, we have

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} \le 1.$$

It is important to observe that $\int_{-\pi}^{\pi} S_m(x)dx = 0$ and that $\sigma = B_m = \sqrt{\frac{1}{2}\sum_{k=1}^{m}(a_k^2 + b_k^2)}$. This indicates that the mean of the series is zero while σ represents its standard deviation. Here n_k denotes a positive integer. For integer sequences, Erdős & Gál (1955) extended the above result considering a particular form and established the corresponding form of LIL. But in 1955, while pursuing her Ph.D., Weiss (1959) established a complete counterpart of Kolmogorov's LIL for lacunary series. Her contribution is regarded as a significant milestone in this field. Remarkably, she achieved this and several other notable results. She obtained:

Theorem 2.4 Let $B_m = \sqrt{\frac{1}{2}\sum_{k=1}^{m}(a_k^2 + b_k^2)}$ and $M_m = \max_{1\le k\le m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$. Assume further that

$B_m \to \infty$ as $m \to \infty$ and S_m fulfills the Kolmogorov-type condition with n_k being integers. Under these assumptions, for a.e. θ, we have

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} = 1.$$

The LIL discussed so far deals with the sum of the first n terms of a given sequence and describes the rate at which this sum grows. This form is commonly known as the regular LIL. In contrast, SZ (Salem & Zygmund, 1950), introduced an alternative version of the LIL that focuses on the tail sums of trigonometric series. Their result is:

Theorem 2.5. Let $S_{\tilde{N}}(\theta) = \sum_{k=N}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ be given lacunary series where $c_k^2 = a_k^2 + b_k^2$ satisfying $\sum_{k=1}^{\infty} c_k^2 < \infty$. Set $B_{\tilde{N}} = \left(\frac{1}{2}\sum_{k=N}^{\infty} c_k^2\right)^{\frac{1}{2}}$ and $M_{\tilde{N}} = \max_{k\geq N}|c_k|$. Assume $B_{\tilde{1}} < \infty$ and that $M_{\tilde{N}}^{2} \leq K_N\left(\frac{B_{\tilde{N}}^{2}}{\mathrm{loglog}\frac{1}{B_{\tilde{N}}}}\right)$ for some $K_N \downarrow 0$ as $N \to \infty$. Then

$$\limsup_{N\to\infty} \frac{S_{\tilde{N}}(\theta)}{\sqrt{2B_{\tilde{N}}^{2}\mathrm{loglog}\frac{1}{B_{\tilde{N}}}}} \leq 1$$

for a.e. θ in the circle of radius one.

This version of the LIL highlights the sums taken after the first n terms, with particular emphasis on the tail sums of the series. Hence it is often known as tail version of the LIL. This version examines the asymptotic behavior of a series beyond a certain index, capturing the characteristics of the tail sums of rapidly oscillating functions. SZ initially obtained the one-sided version of this LIL. Subsequently, under comparable context, Ghimire and Moore (2014) obtained the other version, as stated below:

Theorem 2.6 Using the same notation and assumptions as in the preceding theorem, we have

$$\limsup_{N\to\infty} \frac{S_{\tilde{N}}(\theta)}{\sqrt{2B_{\tilde{N}}^{2}\ln\ln\frac{1}{B_{\tilde{N}}}}} \geq 1$$

for a.e. θ in the circle of radius one.

The results of SZ were later extended to summations involving signum functions, demonstrating the robustness of the tail LIL phenomenon across various analytic settings. Results of Signum function, denoted by sgn(x), are defined as $\mathrm{sgn(x)} = 1$ if $x > 0$, $\mathrm{sgn(x)} = -1$ if $x < 0$ and $\mathrm{sgn(x)} = 0$ if $x = 0$. In this direction, Ghimire (Ghimire, 2022) established a LIL for summation involving signum functions, stated as follows:

Theorem 2.7: Consider signum functions $\{u_i\}$ where $u_i(x) = \mathrm{sgn}\,(\sin 2^i\pi x)$ and let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers with $\sum_{k=1}^{\infty} a_k^2 < \infty$. Then

$$\limsup_{m\to\infty} \frac{\left|\sum_{i=m+1}^{\infty} a_i u_i(x)\right|}{\sqrt{2\ \sum_{i=m+1}^{\infty} b_i^2\ \log\log\left(\frac{1}{\sum_{i=m+1}^{\infty} b_i^2}\right)}} \leq 1$$

for almost every $x \in [0, 1)$.

A lower bound result for the signum function is given by Ghimire (2025).

Theorem 2.8: Consider signum functions $\{u_i\}$ where $u_i(x) = \mathrm{sgn}\,(\sin 2^i\pi x)$ and $\{b_i\}_{i=1}^{\infty}$ with $b_i \in \mathbb{R}$ (set of real numbers) with $B_m = \sum_{i=m}^{\infty} b_i^2$ and assume that $\lim_{m\to\infty} \frac{b_m^2}{B_m} = 0$. Then

$$\limsup_{m\to\infty} \frac{\left|\sum_{i=m}^{\infty} b_i u_i(x)\right|}{\sqrt{2\ B_m \log\log\left(\frac{1}{B_m}\right)}} \geq 1$$

for almost every $t \in [0, 1)$.

Finally, we recall Borel–Cantelli Lemma (Royden & Fitzpatrick, 2010), an essential tool that will be applied in establishing our main result.

Lemma 2.9: Consider a sequence $\{E_k\}_{k=1}^{\infty}$ of measurable sets satisfying $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then, except for a set of measure zero, for point $x \in \mathbb{R}$ lies in only finitely many of E_k.

Thus, we see that there are number of results developed on the LIL in the various contexts of mathematics and statistics. These limit laws are needed in the study of the long-term behavior of various functions representing real life situations. Motivated by these limit laws, we propose and derive a LIL for stopping time indexed sums of signum function.

RESULTS

In this section, we establish a LIL for the tail sums of stopping-time indexed signum function as our main result. We obtain a precise upper bound for this tail LIL. The proof of our main theorem relies on the stopping-time argument, Lemma 2.9, and the continuity property of measure. An alternative derivation of this stopping-time indexed limit law can also be obtained through the dyadic martingale framework (Ghimire, 2022). Our main result is as follows:

Theorem 3.1: Let $g_n(x) = \sum_{k=1}^{n} b_k u_k(x)$ where $\{b_k\} \subseteq \mathbb{R}$ satisfying $\sum_{k=1}^{\infty} b_k^2 < \infty$ and $\{u_k\}$ is a sequence of signum functions. Define stopping time as

$$\eta_k = min\left(m: \quad \sum_{k=m+1}^{\infty} b_k^2 < \frac{1}{\theta^k}, \qquad \theta > 1\right).$$

Then for this stopping time,

$$\limsup_{k\to\infty} \frac{\left|\sum_{k=\eta_k+1}^{\infty} b_k u_k(x)\right|}{\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \, \log\log\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}} \le 1$$

for a.e. x.

DISCUSSION

In this section, we prove our main result (Theorem 3.1).

We note that for each i, the function $E_i(x) = b_i u_i(x)$ is a symmetric and independent random variable with mean 0 and variance 1. Applying Levy's inequality, we have

$$\left|\left\{x \in I : \left|\max_{0\le j\le m} \sum_{i=1}^{j} E_i(x)\right| > \lambda\right\}\right| \le 2 \left|\left\{x \in I : \left|\sum_{i=1}^{m} E_i(x)\right| > \lambda\right\}\right|.$$

Let M be sufficiently greater than m. Then for this M, we have

$$\left|\left\{x \in I : \left|\max_{0\le j\le M-m-1} \sum_{i=0}^{j} E_{M-i}(x)\right| > \lambda\right\}\right| \le 2 \left|\left\{x \in I : \left|\sum_{i=0}^{M-m-1} E_{M-i}(x)\right| > \lambda\right\}\right|$$

But $E_i(x) = g_i(x) - g_{i-1}(x)$ and $g_k(x) = \sum_{i=1}^{k} b_i u_i(x)$. Then we have

$$\left|\left\{x \in I : \left|\max_{M-1\ge n\ge m} g_M(x) - g_n(x)\right| > \lambda\right\}\right| \le 2\,|\{x \in I : |g_M(x) - g_m(x)| > \lambda\}|\,.$$

Observing that $\sup_{k} |a_k| > \lambda$ implies either $\left|\sup_{k} a_k\right| > \lambda$ or $\left|\sup_{k} (-a_k)\right| > \lambda$, we obtain

$$\left|\left\{x \in I : \max_{M\ge n\ge m} |g_M(x) - g_n(x)| > \lambda\right\}\right| \le 2\,|\{x \in I : |g_M(x) - g_m(x)| > \lambda\}| \ldots\ldots\ldots..(1)$$

We recall an inequality:

$$\left|\left\{x \in I : \sup_{n\ge m} \left|\sum_{k=m+1}^{n} b_i u_i(x)\right| > \lambda\right\}\right| \le 6 \ \exp\left(\frac{-\lambda^2}{2\sum_{k=m+1}^{\infty} b_k^2}\right)$$

The proof of the above inequality can be found in Ghimire (2022).

This can be expressed as:

$$\left|\left\{x \in I : \sup_{n\ge m} |g_n(x) - g_m(x)| > \lambda\right\}\right| \le 6 \ exp\left(\frac{-\lambda^2}{2\sum_{k=m+1}^{\infty} b_k^2}\right) \ldots\ldots..(2)$$

From (1) and (2), we get

$$\left|\left\{x \in I : \sup_{M\ge n\ge m} |g_M(x) - g_n(x)| > \lambda\right\}\right| \le 12 \ exp\left(\frac{-\lambda^2}{2\sum_{k=m+1}^{\infty} b_k^2}\right)$$

Next, we define $E_M = \left\{x \in I: \sup_{M\ge n\ge m} |g_M(x) - g_n(x)| > \lambda\right\}$ and $E = \cup_{M=1}^{\infty} E_M$. Clearly $E_M \subseteq E_{M+1}$. Using the continuity property of measure, we have

$$|E| = \lim_{M\to\infty} |E_M|.$$

One can prove that for $g(x) = \sum_{i=1}^{\infty} b_i u_i(x)$,

$$\left\{x \in I : \sup_{\mathrm{M} \geq \mathrm{n} \geq \mathrm{m}} |g_M(x) - g_n(x)| > \lambda\right\} \subseteq E$$

and so,

$$\left|\left\{x \in I : \sup_{\mathrm{n} \geq \mathrm{m}} |g(x) - g_n(x)| > \lambda\right\}\right| \leq 12 \ \exp\left(\frac{-\lambda^2}{2\sum_{k=m+1}^{\infty} b_k^2}\right)$$

This gives

$$|\{x \in I : |g(x) - g_m(x)| > \lambda\}| \leq 12 \ \exp\left(\frac{-\lambda^2}{2\sum_{k=m+1}^{\infty} b_k^2}\right)$$

For the particular case of stopping time η_k, we have

$$\left|\left\{x \in I : \left|g(x) - g_{\eta_k}(x)\right| > \lambda\right\}\right| \leq 12 \ \exp\left(\frac{-\lambda^2}{2\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)$$

Next, we choose $\lambda = (1+\epsilon)\sqrt{\frac{2}{\theta^k}\ln\ln(\theta^k)}$ where $\theta > 1$. This gives

$$\left|\left\{x \in I : \left|g(x) - g_{\eta_k}(x)\right| > (1+\epsilon)\sqrt{\frac{2}{\theta^k}\ln\ln(\theta^k)}\right\}\right| \leq 12 \ \exp\left(\frac{-(1+\epsilon)^2 \ \frac{2}{\theta^k} \ \ln\ln(\theta^k)}{2\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)$$

Using the definition of stopping time, we have

$$\sum\nolimits_{k=\eta_k+1}^{\infty} b_k^2 < \frac{1}{\theta^k}.$$

Using this, we have

$$\left|\left\{x \in I : \left|g(x) - g_{\eta_k}(x)\right| > (1+\epsilon)\sqrt{\frac{2}{\theta^k}\ln\ln(\theta^k)}\right\}\right| \leq 12 \ \exp\left(\frac{-(1+\epsilon)^2 \ \frac{2}{\theta^k} \ \ln\ln(\theta^k)}{\frac{2}{\theta^k}}\right)$$

$$= 12\exp(-(1+\epsilon)^2 \ \ln\ln(\theta^k))$$

$$= \ 12\ (\mathrm{k}\ln\theta)^{-(1+2\epsilon+\epsilon^2)}$$

This is true for all η_k. Summing over all η_k^s, we have

$$\sum_{k=1}^{\infty}\left|\left\{x \in I : \left|g(x) - g_{\eta_k}(x)\right| > (1+\epsilon)\sqrt{\frac{2}{\theta^k}\ln\ln(\theta^k)}\right\}\right| \leq 12\sum_{k=1}^{\infty} \ (\mathrm{k}\ln\theta)^{-(1+2\epsilon+\epsilon^2)} < \infty$$

Applying Lemma 2.11, we find that for almost every x, there exists a sufficiently large M (depending on x) such that for all $k \geq M$, we have

$$\left|g(x) - g_{\eta_k}(x)\right| \leq (1+\epsilon)\sqrt{\frac{2}{\theta^k}\ln\ln(\theta^k)} = (1+\epsilon)\sqrt{\theta}\sqrt{\frac{2}{\theta^{k+1}}\ln\ln(\theta^k)} \ldots (3)$$

Let x be fixed. Using the definition of stopping time, we have

$$\sum\nolimits_{k=\eta_k+1}^{\infty} b_k^2 < \frac{1}{\theta^k}.$$

This gives

$$\theta^k < \frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}.$$

But $\eta_k < \eta_{k+1}$. So we have

$$\sum_{k=\eta_k+1}^{\infty} b_k^2 \geq \frac{1}{\theta^{k+1}}$$

Thus we have

$$\frac{1}{\theta^{k+1}} \leq \sum_{k=\eta_k+1}^{\infty} b_k^2 < \frac{1}{\theta^k} \qquad \ldots\ldots\ldots (4)$$

Using (4) in (3), we get

$$\left|g(x) - g_{\eta_k}(x)\right| \leq (1+\epsilon)\sqrt{\theta}\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \ \ln\ln\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}$$

So for almost every x, we get

$$\frac{\left|g(x) - g_{\eta_k}(x)\right|}{\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \ \ln\ln\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}} \leq \sqrt{\theta}(1+\epsilon).$$

This can be done for all $\epsilon > 0$. This gives

$$\frac{\left|g(x) - g_{\eta_k}(x)\right|}{\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \ \ln\ln\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}} \leq \sqrt{\theta}.$$

Finally letting $\theta \to 1$, we get

$$\frac{\left|g(x) - g_{\eta_k}(x)\right|}{\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \ \ln\ln\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}} \leq 1.$$

We note that as $m \to \infty$ then $k \to \infty$. Then

$$\limsup_{k\to\infty} \frac{\left|g(x) - g_{\eta_k}(x)\right|}{\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \ \ln\ln\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}} \leq 1.$$

Finally we have

$$\limsup_{k\to\infty} \frac{\left|\sum_{k=\eta_k+1}^{\infty} b_k u_k(x)\right|}{\sqrt{2\sum_{k=\eta_k+1}^{\infty} b_k^2 \ \ln\ln\left(\frac{1}{\sum_{k=\eta_k+1}^{\infty} b_k^2}\right)}} \leq 1.$$

This completes the proof the theorem.

CONCLUSION

The present work establishes a LIL for tail sums of stopping-time indexed signum functions. An explicit upper bound for this limit law is obtained using the techniques of stopping-time, Borel–Cantelli lemma, and continuity properties of measure. Our result extends existing LIL for tail sum frameworks and provides further insight into the asymptotic behavior of dependent structures indexed by random times.

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AUTHORS CONTRIBUTION

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CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

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The authors confirm that this manuscript is original, has not been published previously, and is not under consideration for publication elsewhere.

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No data were used in the article.

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None

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