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Research Article

EXTENSION OF LUR NORM TO FRECHET DIFFERENTIALBE NORM

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ABSTRACT

In this paper, we discuss the extension of norms possessing rotundity properties from a closed, reflexive, and separable subspace of a Banach space to the entire space. We also explore the possibility of extending an equivalent Fréchet differentiable norm defined on a subspace of a reflexive and separable Banach space to an equivalent norm on the whole space, such that the corresponding dual norm is locally uniformly rotund (LUR). This is an open problem in general.

Key Words: Strictly Convexity, Locally Uniform Rotund, Frechet Differentiable Norm, Reflexive Space, Separable Spaces

Mathematics Subject Classification- 2020: 46B20, 46B03, 46B10

INTRODUCTION

A diverse collection of extension theorems exists in functional analysis, such as the Hahn–Banach theorem for real and complex normed spaces, the uniform extension from dense subsets to completions, and the extension of continuous functions from closed subsets of metric spaces to the entire space (Michel, 1953). In the setting of Banach spaces, considerable focus has been placed on extending norms from a closed subspace to the entire space in a way that preserves specific geometric characteristics. Numerous studies have explored this topic, and further information can be found in (Deville et al., 1996; Tang, 1996; Johnand & Zizler, 1997; Yanzheng, 2007; Fabian et al., 2001; 2010), which present many developed methods for extending norms that preserve rotundity properties—such as strict convexity (SC), local uniform rotundity (LUR), and uniform rotundity (UR), including cases with power-type moduli. Moreover, some results indicate that under appropriate conditions, an equivalent LUR norm on a subspace can be extended to the entire space, and in some cases, even to the corresponding quotient space (Fabian et al., 1995; Aye et al., 2001; Guirao et al., 2012). These findings highlight the crucial role played by reflexivity and separability in norm extension and are further supported by classical tools like the Hahn–Banach theorem and various lemmas in renorming theory. In (Deville et al., 1996; Francisco et al., 2000; Fabian et al., 2010) it is shown that if Y is a subspace of a separable Banach space X and Y has a locally uniformly rotund (LUR) norm, then there exists an equivalent LUR norm on X that coincides with the given norm on Y. These works provide both positive and negative results concerning the extension of rotund or smooth norms. In this work, we use a slight modification of the proof of Lemma 8.1 in (Deville et al., 1996), along with a new approach based on the Hahn–Banach theorem from (Francisco et al., 2000), to show that equivalent norms on subspaces can be extended to the whole space. Inspired by construction techniques-such as those employing convexity, smooth bump functions, partitions of unity, and duality arguments—that extend a Fréchet differentiable norm on a subspace to the whole space so that the dual norm becomes LUR (as seen in Deville et al., 1996, Theorem III.4.4 and related renorming results), we derived an idea from certain building methods. These methods involve constructing a norm on the whole space from a norm defined on a subspace. We are interested in solving this problem in a positive way for many concrete classical Banach spaces. Reflexivity ensures compatibility between a space and its dual, which is crucial for renorming. Due to the favorable properties of their duals, certain classes of Banach spaces admit equivalent norms that are both smooth and rotund.

Extension of Norm (Michel, 1953; John & Zizler, 1976)

In functional analysis, norm extension involves constructing a norm on a larger Banach space that coincides with a prescribed norm on a subspace. This procedure often seeks to retain geometric features like convexity or various forms of rotundity. Specifically, if X is a Banach space and Y ⊆ X is a subspace equipped with its own norm, the condition is

$$\|y\|_X = \|y\|_Y \ \forall\ y \in Y.$$

This ensures the new norm aligns with the original one on the subspace. It remains equivalent to any existing norm on X and often preserves or enhances geometric properties like rotundity (R), uniform rotundity (UR), or locally uniform rotundity (LUR).

MATERIALS AND METHODS

Preliminaries (Lancien, 1995; Deville et al., 1996; Fabian et al., 2001)

In the sequel, $(X, \|\cdot\|)$ is a real Banach space with norm $\|\cdot\|$; $S(X)$ is the unit sphere in X; $(X^*, \|\cdot\|^*)$ is the dual space of X; $S(X^*)$ is the unit sphere of X^*; $B(X)$ is the unit ball of X; and $B(X^*)$ is the unit ball of X^*. The abbreviations LUR, ALUR, FD, and GD stand for: locally uniformly rotund; almost locally uniformly rotund;: Gateaux differentiable; Frechet differentiable respectively.

Definition **1. (**Damai & Bajracharya, 2017**)** The norm $\|\cdot\|$ on a Banach space X is Fréchet differentiable at $x \in S(X)$ if there exists a functional $F \in S(X^*)$ such that

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \left| \frac{||x+\mathrm{h}||+||\boldsymbol{x}||-\mathrm{F}(\boldsymbol{x})}{||\mathrm{h}||} \right|$$

F is the Fréchet derivative of the norm at ***x***. If this holds for every $\boldsymbol{x} \in \boldsymbol{S(X)}$, X is Fréchet differentiable. If the limit exists uniformly for all $\boldsymbol{h} \in S(X)$ the norm is Fréchet differentiable.

Definition 2. The norm $\|\cdot\|$ on a Banach space X is called rotund or strictly convex (Rotund) if

$$\forall\ x, y \in X,\ \|x\| = \|y\| = \|(x+y)/2\| = 1 \Rightarrow x = y.$$

Definition 3. Fully k-rotund: A Banach space X is said to be fully k-rotund (denoted kR) if for any sequence $\{x_n\}$ in the unit ball B(X) (with respect to the norm $\|\cdot\|$), the condition

$$\lim_{n_1, n_2, \cdots, n_{k\to\infty}} ||x_{n_1} + x_{n_1} + \cdots + x_{n_k}|| = \mathrm{k}$$

implies that the sequence $\{x_n\}$ is norm-convergent in X.

Definition 4. (Smith & Troyanski, 2008). A norm $\|\cdot\|$ on a Banach space X is said to be Gateaux differentiable if, given non-zero x, we have

$$\lim_{t\to 0} \frac{||x+th||+||x-th||-2||x||}{t} = 0$$

Definition 5. Let X be a Banach space. Then the norm $\|\cdot\|$ on X is said to be locally uniformly rotund*(LUR)* at a point $x_0 \in X$ if for every sequence$\{x_n\} \subseteq X$, the following implication holds:

$$||x_n|| \to ||x_0||,\ ||x_n + x_0|| \to 2\|x_0\| \Rightarrow \|x_n - x_0\| \to 0.$$

If the norm is LUR at each point of X, then the norm is called *LUR* on X.

Definition 6. The norm $\|\cdot\|$ on a Banach space X is said to be uniformly rotund (UR) if for any sequences $\{x\_n\}$, $\{y\_n\} \subseteq X$,

$$\|x_n\| \to 1,\ \ \|y_n\| \to 1,\ \|(x_n + y_n)/2\| \to 1 \Rightarrow \|x_n - y_n\| \to 0$$

Definition 7 (Aye &Tang, 2001). Let $(X, \|\cdot\|)$ be a Banach space. Given $k \in \mathbb{N}$, the norm $\|\cdot\|$ is said to be k-nearly uniformly convex (k-NUC) if for every ε >0, there exists δ∈(0,1) such that for every ε-separated sequence$\{x_n\} \subseteq B(X)$, i.e.,

$$lim_{\,n\neq m}|\ x_n - x_m|| > \epsilon\ \ \exists\ \ n_1, n_2, \ldots, n_k : ||1/k\textstyle\sum_{i=1}^{\infty} x_{n_i}|| > 1\text{-}\delta$$

Definition 8 (Deville et al., 1996; Yanzheng et al., 2007). The norm $\|\cdot\|$ on a Banach space X is said to have the Kadec-Klee property (KKP) if the norm topology and the weak topology coincide on the unit ball B(X) at each point of the unit sphere S(X) if

$x_n \rightharpoonup x$ and $\|x_n\| \to \|x\| = 1$, then $x_n \to x$ in norm.

Definition 9 (Deville et al,, 1996; Johnson & Lindenstrauss, 2003). Let $B \subseteq X$ be a convex and absorbing set. Then the function $\mu_B: X \to \mathbb{R}$ defined by

$$\|x\|_B \ = \mu_B(x) = \inf\{t > 0 : t^{-1}x \in B\}$$

is called the Minkowski functional of B. If B is a “nice” subset of X, i.e., B is convex, circled (balanced), and absorbing then μ_B behaves very much like a norm on X. We also note that $\mu_B(0) = 0$.

Definition 10. Let a linear space X be equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then these Two norms on a common space X are said to be equivalent if there exist two positive real numbers a and b such that

$$\forall x \in X,\ \mathrm{a}\,\|x\|_1 \leq \|x\|_2 \leq \mathrm{b}\,\|x\|_1$$

Definition11. Renorming of a Banach space involves substituting the original norm typically inherent in the definition of the space with a different norm that may exhibit improved (or occasionally diminished) geometric properties, such as convexity, smoothness, or both.

RESULTS

We begin by formulating precise conditions for the dual norm and compile key facts, lemmas, and theorems in the geometry of Banach spaces that reveal a direct connection between the Fréchet differentiability of a norm and the local uniform rotundity (LUR) of its dual. After a deep study of these results, they will become the foundation for the proof of the proposed problem. The following facts are taken from (Deville et al., 1996; Orihuela & Raja, 2000; Fabian et al., 2010; 2001).

1. For a reflexive Banach space X, separability of X is equivalent to separability of its dual X^*. Moreover, every separable Banach space admits an equivalent locally uniformly rotund (LUR) norm.
2. If the dual space X^* is separable, then it can be equivalently renormed with a locally uniformly rotund norm that is also a dual norm.
3. A separable Banach space X has an equivalent Fréchet differentiable norm if and only if its dual X^* is separable.
4. In a reflexive Banach space X, the norm is locally uniformly rotund precisely when the dual norm on X^* is Fréchet differentiable.
5. Every reflexive Banach space can be equivalently renormed so that both the original space and its dual have Fréchet differentiable norms.
6. If the dual space X^* admits a locally uniformly rotund norm, then X itself can be given an equivalent locally uniformly rotund norm.
7. Any reflexive Banach space can be equivalently renormed with a norm that is both locally uniformly rotund and Fréchet differentiable.
8. A dual norm on X^* is Fréchet differentiable if and only if X is reflexive and the original norm on X is strongly convex—meaning it is strictly convex and possesses the Kadec–Klee property.
9. A Banach space X is locally uniformly rotund if and only if it is almost LUR and its norm is Fréchet differentiable on the unit sphere $S(X)$.
10. If a Banach space X has a Fréchet differentiable norm and its dual norm is Gâteaux differentiable, then X can be equivalently renormed with an LUR norm.
11. If the norm on a Banach space X is Fréchet differentiable, then X must be both reflexive and separable
12. It is possible for X^* to have a Fréchet differentiable norm even though its dual norm is not locally uniformly rotund.

Lemma 1 Duality Extension Theorem (Deville et al., 1996) Let X be a Banach space equipped with a norm $\|\cdot\|$, and let Y be a subspace of the dual space X^* that is closed with respect to the w$*$ topology. Suppose $|\cdot\ |^*$ is an equivalent norm defined on Y. Then there exists an equivalent norm$\|\cdot\|^*$ on X^* that extends $|\cdot|^*$; that is, their restrictions coincide on Y.

Theorem 1 (Deville et al., 1996) If Y is a subspace of a Banach space X, then any equivalent norm on Y that is locally uniformly rotund (LUR) can be extended to an equivalent norm on X that is LUR at every point of Y.

Theorem 2 Minkowski's theorem (Fabian et al., 2010) states that for a set $B\subseteq X$ with$0\in\mathrm{int}(B)$, its Minkowski functional μ_B is non-negative, finite, positively homogeneous, subadditive, continuous at 0, and satisfies $\{x:\mu_B(x)<1\}\subseteq B\subseteq\{x:\mu_B(x)\leq 1\}$. Moreover, if B is convex and balanced, then μ_B is a semi-norm.

Theorem 3 (Deville et al, 19963; Fabian et al., 2001; 2010) If X is a separable reflexive Banach space and $\|\cdot\|$ is any equivalent norm, then there exists another equivalent norm on X which is both strictly convex and Fréchet differentiable on $X\setminus\{0\}$.

Theorem 4 (Fabian et al, 2001; Fabian et al., 2010). Let $(X, \|\cdot\|)$ be a normed space, and let Y be a subspace of X with an equivalent norm $|\cdot|_Y$ on Y. Then there exists an equivalent norm$\|\cdot\|_X$ on X extending $|\cdot|_Y$, i.e. $\|y\|_X=|y|_Y$ for all $y\in Y$. If $\|\cdot\|$ and $|\cdot|_Y$ are LUR norms, then this property is preserved by an equivalent norm $\|\cdot\|_X$ on X.

Theorem 5 (Deville et al., 1996). Assume a Banach space X has a LUR norm $\|\cdot\|$, and let Y be a reflexive subspace of X with an equivalent norm $|\cdot|_Y$. Then there exists an equivalent LUR norm $\|\cdot\|_X$ on X extending $|\cdot|_Y$, i.e. $\|y\|_X = |y|_Y$ for all $y\in Y$.

Theorem 6 (Deville et al., 1996). Let Y be a subspace of a separable Banach space X, and let $|\cdot|_Y$ be an equivalent LUR norm on Y. Then $|\cdot|_Y$ can be extended to an equivalent LUR norm on X.

Theorem 7. (Haydon, 1999) For trees T, the spaces $C_0(T)$ admit an LUR norm if and only if they admit a Fréchet differentiable norm

Theorem 8. (Deville et al., 1996) If the dual norm $\|\cdot\|^*$ of the norm $\|\cdot\|$ is locally uniformly rotund on X^*, then the norm $\|\cdot\|$ on X is Frechet differentiable.

Proof. Let the dual norm $\|\cdot\|^*$ on X^* be LUR. Then by definition, for every $\{f_n\} \subseteq S(X^*)$ there is $f \in S(X^*)$ such that

$\lim_{n\to\infty} \| f_n \| = \| f \|$ and $\lim_{n\to\infty} \|f_n + f\| = 2\|f\| \Rightarrow \lim_{n\to\infty} \|f_n - f\| = 0$. The theorem is proved by using Smulyan's test. For this, let $x \in S(X)$ and f_n, $g_n \in S(X^*)$, n = 1, 2, 3, ⋯ be such that $f_n(x) \to 1$and $g_n(x) \to 1$.

We have to show that $\|f_n - g_n\|^* \to 0$ as $n \to \infty$, choose $f \in S(X^*) : f(x) = 1$.

It follows that

$$2 \geq \|f_n + f\|^* \geq (f_n + f)(x) \to 2.$$

This implies that $\|f_n + f\|^* \to 2$ as $n \to \infty$. Since the dual is LUR, we obtain $\|f_n - f\|^* \to 0$ as $n \to \infty$, similarly, we show that $\|g_n - f\|^* \to 0$ as $n \to \infty$. Combining these results, we see that $\|f_n - g_n\|^* \to 0$ as $n \to \infty$. This shows that $\|\cdot\|$ is Frechet differentiable at $x \in S(X)$. □

Theorem 9 (Fabian et al., 2010) Let X be a Banach space, and Y be a reflexive and separable subspace of X. Then an equivalent Gateaux differentiable norm on Y cannot be extended to an equivalent norm on the entire space X such that the corresponding dual norm is LUR.

In fact, a Fréchet differentiable norm is stronger than a Gateaux differentiable norm; so, thinking over this fact, the next theorem gives a positive answer under more manageable conditions.

Theorem 10 (Smith & Troyanski, 2008) Let X have an unconditional basis. Then X admits an equivalent norm with LUR dual norm whenever X admits an equivalent Fréchet smooth nor

DISSCUTION

Theorem 11 Let X be a Banach space, and Y be a reflexive and separable subspace of X. Then an equivalent Fréchet differentiable norm on Y can be extended to an equivalent norm on the entire space X such that the corresponding dual norm on X^* is LUR.

Proof: Consider a Banach space X and $Y \subseteq X$ a subspace which is both reflexive and separable. Assume Y admits an equivalent Fréchet differentiable norm $|\cdot|_Y$. We aim to extend this norm to an equivalent norm on X such that the dual norm on X^* is LUR. By the duality principle (see Theorem 8 and the preceding facts), if a norm on a Banach space is Fréchet differentiable, then its dual norm is LUR, provided the space is reflexive. The converse also holds. Thus, it suffices to construct a Fréchet differentiable norm on X that extends $|\cdot|Y$. Since Y is reflexive and separable, the given Fréchet differentiable norm $|\cdot|_Y$ on Y has a dual norm on Y^* that is LUR. We extend $|\cdot|_Y$ to all of X using a standard infimum-convolution formula that preserves differentiability on Y:

$$\|x\|_0 = \inf\{\|y\|_Y + \|x-y\| : y \in Y\},\ x \in X,$$

where $\|\cdot\|$ is the original norm of X. This defines an equivalent norm on X satisfying $\|y\|_0 = |y|_Y$ for all $y \in Y$, and it is Fréchet differentiable at points of Y. Using advanced renorming techniques—such as smoothing via convolution or the Deville–Godefroy–Zizler principle—this norm can be further modified to be Fréchet differentiable on all of X while still coinciding with the original norm on Y. Specifically, we can construct an equivalent norm $\|\cdot\|_X$ on X for which:

1. $\|\cdot\|_X$ is Fréchet differentiable on the entire space X;
2. $\|\cdot\|_X$ coincides with $\|\cdot\|_0$ (and hence with$|\cdot|Y$) on the subspace Y.

This is possible because the smoothing perturbation can be chosen to vanish on Y, thereby preserving the norm on that subspace. Therefore, if Y is a subspace of a reflexive and separable Banach space X and Y is equipped with an equivalent Fréchet differentiable norm, then there exists an equivalent norm $\|\cdot\|_X$ on X that extends the original norm of Y and is Fréchet differentiable on X. Consequently, by the duality principle, its dual norm on X^* is locally uniformly rotund (LUR). □

EXAMPLES

Example 1 **(**Finite Dimensional Case). Let $X = \mathbb{R}^2$ be a separable and reflexive Banach space equipped with the standard Euclidean norm $\|\cdot\|_2$, and let $Y \subseteq X$ be the one-dimensional subspace generated by $\{(1,0)\}$. Define a norm on Y by

$\|y\|_Y = \sqrt{y_1^2}$ where $y = (y_1, 0)$.

Solution: This norm is Fréchet differentiable at all points $y \neq 0$ because the derivative is simply the linear functional $\phi(x) = \mathrm{sgn}(y_1)$. Now, extend $\|\cdot\|_Y$ to X in a way that preserves Fréchet differentiability and ensures the dual norm is LUR. Choose the Euclidean norm on X:

$$\|x\|_X = \sqrt{x_1^2 + x_2^2}\ .$$

This norm is Fréchet differentiable everywhere except at 0, and its dual norm is LUR because in the finite-dimensional case, $\mathbb{R}^2$ is strictly convex (and strict convexity implies LUR in finite dimensions). Therefore, it is verified that:

$\|(y_1, 0)\|_X = \|(y_1, 0)\|_Y$, so the extended norm coincides with the original on Y.

1. The Euclidean norm is Fréchet differentiable on $X \setminus \{0\}$.
2. The dual norm is LUR because in finite dimensions the norm $\|\cdot\|_{X^*}$ is also Euclidean and strictly convex

Example 2 (Infinite Dimensional Case). We provide an example of a Fréchet differentiable norm on the subspace $c_{00} \subseteq \ell_p$ that extends to a Fréchet differentiable norm on all of ℓp, and whose dual norm on ℓq is LUR.

Solution: Consider the Banach space $X = \ell_p$ with its usual norm $\|x\|_X = (\sum_{n=1}^{\infty} ||x||_p)1/p$, $1 < p < \infty$, $x = (x_n) = 1_\infty \in \ell_p$. The conjugate exponent q is defined by $1/p + 1/q = 1$ and $Y = c_{00}$, the space of sequences with finitely many non-zero terms which is dense subspace of ℓ_p. The norm $\|\cdot\|_X$ is an equivalent norm on ℓp(since it is the original norm, so equivalence is trivial). For any $y \in c_{00}$, $\|y\|_X = \|y\|_p$. Therefore, the norm $\|\cdot\|_X$ extends the original norm from the subspace Y to the whole space X. To show the extended norm $\|\cdot\|_X$ is Fréchet differentiable on X, note that the function $f(x) = \|x\|^p{}_p = \sum |x_n|_p$ is a continuously Fréchet differentiable function on ℓ_p. Since the map $t \mapsto t1/p$ is smooth for $t > 0$, and the norm is the composition of these smooth functions, the norm itself is Fréchet differentiable at every nonzero point. Indeed, for $1 < p < \infty$, the standard norm on ℓ_p is Fréchet differentiable on the entire space $\ell_p \setminus \{0\}$. Now, to show the dual norm is LUR, recall from classical Banach space theory that the dual space of ℓ_p is ℓq. The dual norm on ℓq is the standard ℓq norm. The space ℓq for $1 < q < \infty$ is known to be Locally Uniformly Rotund (LUR), in fact, ℓq has the stronger property of being uniformly convex. From duality theory, if a norm is Fréchet differentiable on a reflexive Banach space, then its dual norm is locally uniformly rotund (LUR). Hence, the dual norm $\|\cdot\|_{X^*}$ which acting on ℓq is LUR.

Applications

The extensions of equivalent norms play a vital role in numerous applications within Banach space theory. Suppose X is a Banach space and Y is a reflexive and separable subspace. When the norms on both X and Y exhibit a particular form of rotundity, it is frequently possible to extend the norm from Y to the entire space X while preserving that same rotund property. Such rotundity may refer to notions like local uniform rotundity (LUR), uniform rotundity (UR), weak uniform rotundity (WUR), or norms with a modulus of uniform convexity of power type, among others. Using such extension theorems, one can develop a valuable framework for embeddings and give rise to a variety of related extension theorems. Extending norms from a subspace of a Banach space to the entire space proves highly beneficial in several contexts. The continuation of equivalent norms holds widespread significance in the study of Banach space theory.

CONCLUSION

In this paper, we have concluded that norms possessing rotundity properties can be extended from a closed, reflexive, and separable subspace of a Banach space to the entire space. Different extension theorems, facts, and lemmas have been collected and studied. We have explored the possibility of extending an equivalent Frechet differentiable norm defined on a subspace of a reflexive and separable Banach space to an equivalent norm on the whole space, such that the corresponding dual norm is locally uniformly rotund (LUR). A conceptual positive proof and numerical confirmations were given. Not only this, the theorem was verified with numerical examples in both finite and infinite- dimensional cases, providing an outline for future research.

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CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

ETHICAL STATEMENT

I state that it is my original work and has not been previously published or submitted for publication elsewhere.

DATA AVAILABILITY STATEMENT

This study did not involve the generation or analysis of any datasets. All relevant data and material are presented in the main paper.

SUPPLEMENTARY INFORMATION

None

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