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Research Article

MULTIPLICATIVE TOPOLOGICAL INDICES OF UNIT GRAPHS ON FINITE COMMUTATIVE RING

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ABSTRACT

We provide and examine the multiplicative atom-bond connectivity index (MABCI) and the multiplicative Randić index (MRI) for unit graphs connected to Commutative rings, specifically for direct products of finite fields. For these indices, we provide explicit combinatorial formulas by applying the H-join decomposition method. A systematic methodology for calculating these indices is established by characterising edge partitions and degree distributions.

Keywords: Commutative ring, Idempotents, Topological indices, Units

Mathematics Subject Classification- Primary: 05C25, 05C92; Secondary: 13A99

INTRODUCTION

Graph-theoretic methods provide a useful way to examine algebraic structures by associating graphs with rings and then studying their invariants. Classical distance- and degree-based indices, such as the Wiener index and related molecular descriptors, have been widely used to describe structural properties of graphs (Wiener, 1947; Gutman & Trinajstić, 1972; Gutman, 1996). These ideas have also influenced the study of algebraic graphs arising from commutative rings.

The zero-divisor graph introduced by Beck (1988) and further developed for commutative and finite rings has shown how algebraic relations can be translated into graph adjacency (Anderson & Livingston, 1999; Redmond, 2002; Akbari et al., 2004). In contrast, the unit graph considered in this work joins two ring elements when their sum is a unit. This additive adjacency condition produces a structure that is different from zero-divisor graphs and requires a separate analysis of degree distributions and edge partitions.

The purpose of this research is to determine explicit combinatorial formulae for the Multiplicative Randić Index (MRI) and the Multiplicative Atom-Bond Connectivity Index (MABCI) of unit graphs associated with direct products of finite fields. The main questions addressed are how the vertices of such unit graphs can be partitioned, how the H-join construction can be used to describe their adjacency, and how these partitions lead to repeatable formulae for the two multiplicative topological indices.

MATERIALS AND METHODS

This study is theoretical and uses algebraic graph construction, vertex partitioning, and combinatorial enumeration. No physical equipment, sampling procedure, or statistical software is involved. The repeatable procedure is to select the commutative ring, construct its unit graph, partition the vertices by their support sets, determine the admissible adjacencies, and then substitute the obtained degree and edge-count data into the required multiplicative indices.

The H-join framework is used as the main construction tool because it allows the unit graph to be described through a base graph together with support-based factor graphs. Published graph-theoretic terminology and the generalized join concept are used with the modifications required by the additive unit condition (Schwenk, 1974; West, 2001; Selvakumar & Gangaeswari, 2022).

We follow (West, 2001) for graph-theoretic concepts.

Unit graph of a commutative ring with unity

Definition 1

Let $U(R) = \{\text{all unit elements in R}\}$. The unit graph $\Omega_U(R)$ is a graph with all ring elements as vertices and two vertices $x, y \in R$ are linked by an edge if and only if their addition is a unit.

Example 1

Consider the ring $R = \mathbb{Z}_5 \times \mathbb{Z}_2$. Clearly this is a commutative ring as all of its elements can be written in the form (x, y) where $x \in \mathbb{Z}_5$ and $y \in \mathbb{Z}_2$. The unit elements of R is $U(R) = \{(1,1), (2,1), (3,1), (4,1)\}$. According to the definition of unit graph, the associated unit graph $\Omega_U(R)$ can be seen in the following Figure 1. Also, the unit graph of $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ is depicted in Figure 2.

Unit graphs as H-join of graphs

The graph generalised join operation is represented by $H\,[G_1, G_2, \ldots, G_m]$. This was initially presented by (Schwenk, 1974), offers a methodical approach to characterizing the adjacency connections in zero divisor graphs of rings. This idea is applied for examining unit graphs in commutative rings with a modular structure. It reflects the unit condition under modular addition aligning with the commutative ring structure of R.

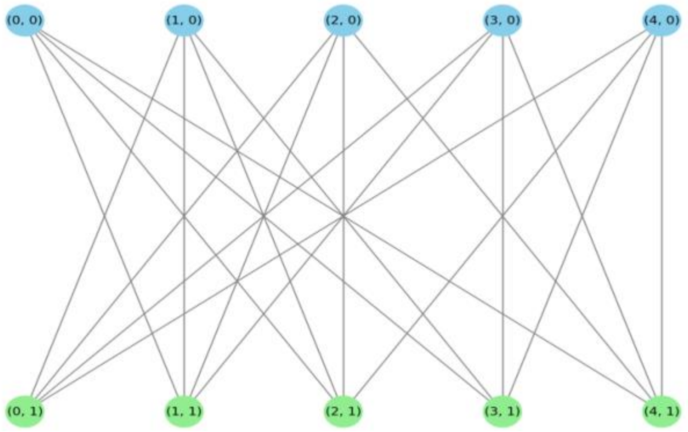

Figure 1. Unit graph associated with $\mathbf{R} = \mathbb{Z}_5 \times \mathbb{Z}_2$

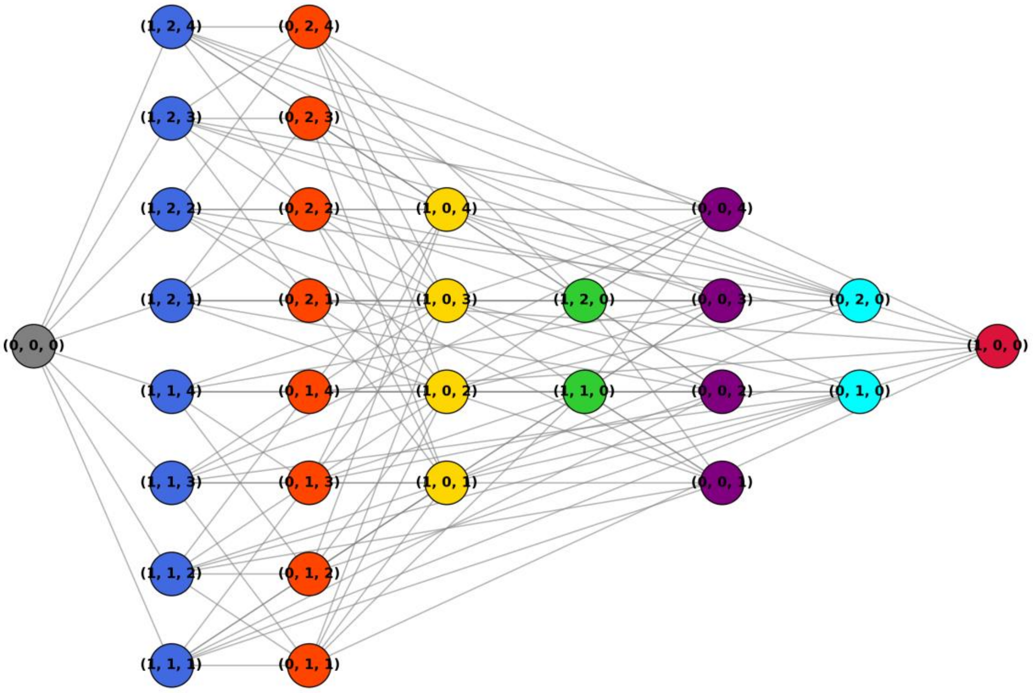

Figure 2. Unit graph for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

Definition 2

Let H be a simple graph with $V(H) = \{1,2,\dots,m\}$. For each $i \in V(H)$, let G_i be a graph on a vertex set $V_i = \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$, with the V_i's pairwise disjoint. The generalized composition (also written $H[G_1,\dots,G_m]$) defining the unit graph $\Omega_U(R)$ is the graph with

$$V(\Omega_U(R)) = \bigsqcup_{i=1}^{m} V_i$$

and edges given by:

1. Intra-part edges: for $i \in \{1,\dots,m\}$, two vertices $x, y \in V_i$ are adjacent in $\Omega_U(R)$ iff they are adjacent in G_i.

2. Inter-part joins: for $i \neq j$, every vertex of V_i is adjacent to every vertex of V_j in $\Omega_U(R)$ iff $ij \in E(H)$.

Equivalently,

$$E(\Omega_U(R)) = \left(\bigcup_{i=1}^{m} E(G_i)\right) \cup \left(\bigcup_{ij\in E(H)} V_i \times V_j\right).$$

Here, H is called the primary graph and $G_1, G_2, \dots, G_m$ are called as the factors of $\Omega_U(R)$. In $R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \dots \times \mathbb{F}_{q_m}$, where $q_i = {p_i}^{m_i}$, the vertex set of $\Omega_U(R)$ is partitioned into subsets B_A based on equivalence classes $A \sqsubseteq \{1,2,\dots,m\}$ determined by the support of elements under modular addition. The base graph H has vertex set $V(H) = \{A | A \in P_m, A \neq \emptyset, A \neq \{1,2,\dots,m\}\}$. Vertices A and B in H are adjacent if $A \sqcup B = \{1,2,\dots,m\}$.For each class B_A, the corresponding factor graph G_A is an independent set as elements within B_A can form a unit under addition modulo $q_i = {p_i}^{k_i}$. Hence the unit graph $\Omega_U(R) = H[G_A : A \in P_m]$.

The unit graph presented in Example 1 for the Commutative ring $\boldsymbol{R = \mathbb{Z}_5 \times \mathbb{Z}_2}$ is the H-join $\boldsymbol{K_2}[\overline{\boldsymbol{K_5}}, \overline{\boldsymbol{K_5}}\,]$. The base graph $\boldsymbol{H = K_2}$ connects the two partitions $\boldsymbol{V_1} = \{(\boldsymbol{a}, \boldsymbol{0})\}$ and $\boldsymbol{V_2} = \{(\boldsymbol{a}, \boldsymbol{1})\}$. Also, the factor graphs $\boldsymbol{G_1} = \overline{\boldsymbol{K_5}}$ and $\boldsymbol{G_2} = \overline{\boldsymbol{K_5}}$ are independent sets as no vertices within $\boldsymbol{V_1} = \{(\boldsymbol{a}, \boldsymbol{0})\}$ or $\boldsymbol{V_2} = \{(\boldsymbol{a}, \boldsymbol{1})\}$ are adjacent.

The unit graph of a commutative ring as generalized join of graphs

For the study of unit graph $\Omega_U(R)$, it is introduced that a support based equivalence relation on the elements of R considering modular addition as the primary operation and adapt the concept of H-join suitable for the adjacency properties unique to unit graphs. First we define an equivalence relation for unit graphs on elements of the Commutative ring R. For $r, s \in R$, define $r \sim s$ if and only if $supp(r) = supp(s)$, where $supp(r) = \{i \,|r_i \neq 0, r = (r_1, r_2, \dots., r_m) \in R\}$. This equivalence relation partitions the elements of R into disjoint subsets B_A where $A \sqsubseteq \{1,2,\dots.,k\}$ and $A \neq \emptyset$. These subsets B_A are called support based equivalence classes for $\Omega_U(R)$.

Next, we assume that $R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \dots \times \mathbb{F}_{q_m}$, where $q_i = {p_i}^{k_i}$ and $\mathbb{F}_{q_i}$ is a finite field of order q_i. For all operations in R we consider addition modulo q_i. Also, define $P_k = \{A \sqsubseteq \{1,2,3,\dots m\}$ with $\emptyset\}$. For $A \in P_k$, let $B_A = \{(r_1, r_2, \dots., r_m) \in R | r_i \neq 0 \iff i \in A\}$ and define a characteristic vector $1_A = (r_1, r_2, \dots., r_m)$ where $r_i = 1$ if $i \in A$ and $r_i = 0$ otherwise.

The vertex set of $\Omega_U(R)$ is $V(\Omega_U(R)) = \bigcup_{A\in P_m} B_A$, where $\bigcup$ is the set of all disjoint union of equivalence classes B_A. Each vertex corresponds to an element of R. Each B_A represents elements in R whise supports correspond to A. These are the sets where addition of any two elements within the same class can result in a unit (that is $r \oplus s \in U(R)$), as their combined supports do not include all indices $\{1,2,3,\dots m\}$.

Two vertices r and s in R are joined by an edge in $\Omega_U(R)$ if and only if the modular sum $r \oplus s \in U(R)$, the supports of r and s must satisfy $supp(r) \sqcup supp(s) = \{1,2,3,\dots m\}$. The sub graph induced by B_A in $\Omega_U(R)$ is a graph on $n_A = \prod_{i\in A}(q_i - 1)$ vertices with $|B_\emptyset| = 1$.

Since the characteristic vector 1_A of a set $A \in P_k$ is the established characteristic of the class B_A and which corresponds to a unique support pattern, the total equivalence classes in $\Omega_U(R)$ is 2^m. Let H be the base graph whose vertices in $V(H) = \{1_A | A \in P_m\}$ represent the equivalence classes in $\Omega_U(R)$. The vertices 1_A and 1_B are linked if and only if $AUB = \{1,2,3,\dots m\}$.

Vertex degrees and edge partitions in unit graph of commutative ring R

Lemma 1

Let $R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \dots \times \mathbb{F}_{q_m}$ where each $\mathbb{F}_{q_i}$ is finite field of order q_i and consider the unit graph $\Omega_U(R)$ with vertex set partitioned into equivalence classes C_A indexed by support sets $A \sqsubseteq \{1,2,\dots.,m\}$. For a vertex $v \in B_A$, the degree $d_{\Omega_U(R)}(v)$ is given by

$$d_{\Omega_U(R)}(v) = \begin{cases} (\prod_{i=1}^{m}(q_i-1))-1, \ \text{if } v\in B_{\{1,2,\dots,m\}} \\ \prod_{j\in B}(q_j-1), \text{if } \ v\in B_A, \ \text{ where } AUB=\{1,2,\dots,m\} \end{cases}$$

Proof. Case 1: A vertex in $B_{\{1,2,\dots,m\}}$ has all its components nonzero. In this case the sum of any two such vertices under modular addition will remain a unit unless their sum modulo q_i results in zero in some coordinates. All vertices in $B_{\{1,2,\dots,m\}}$ are connected within the subset. Since the total vertices in $B_{\{1,2,\dots,m\}}$ is $\prod_{i=1}^{m}(q_i-1)$, each vertex has $d_{\Omega_U(R)}(v) = (\prod_{i=1}^{m}(q_i-1))-1$.

Case 2: A vertex in B_A containing zero components. For a vertex $v\in B_A$ where $A\subset\{1,2,\dots,m\}$ represents the nonzero components. The adjacency condition states that v is adjacent to $u\in B_B$ if and only if $u\oplus v$ is a unit. This occurs when $A\cup B=\{1,2,\dots,m\}$, which means that u must include nonzero components in the exact locations where v is lacking. The number of such neighbors is $d_{\Omega_U(R)}(v)=\prod_{j\in B}(q_j-1)$, where $B=\{1,2,\dots,m\}\backslash A$ is the set of missing indices in v.

Edge partitions

Let $R=\mathbb{F}_{q_1}\times\mathbb{F}_{q_2}\times\dots\times\mathbb{F}_{q_m}$. The unit graph $\Omega_U(R)$ is structured as a H-join $H[G_A: A\in P_m]$, where each subset B_A is a support based partition. The number of edge connections in $\Omega_U(R)$ can be categorized as follows,

(i) The total edges within $B_{\{1,2,\dots,m\}}$ is

$$\left|E_{B_{\{1,2,\dots,m\}}B_{\{1,2,\dots,m\}}}\right| = \left(\prod_{i=1}^{m}(q_i-1)\right)\times\frac{(\prod_{i=1}^{m}(q_i-2)-1)}{2}$$

(ii) The total edges between $B_{\{1,2,\dots,m\}}$ and B_i is

$$\left|E_{B_{\{1,2,\dots,m\}}B_i}\right| = \left(\prod_{i=1}^{m}(q_i-1)\right)\times\left[\left(\sum_{j=1}^{m}\prod_{k\neq j}q_k\right)-4\left(\sum\nolimits_{j=1}^{m}q_j\right)+10\right]$$

(iii) The total edges between B_i and B_j is

$$\left|E_{B_iB_j}\right| = \left(\prod_{i=1}^{m}(q_i-1)\right)\times\left(\sum\nolimits_{j=2}^{m}(q_j-2)+1\right)$$

The MABCI and MRI

The derivation of these indices follows from the vertex and edge partitions given in the preceding section.

The multiplicative Randic index (Gutman et al., 2018) is defined as

$$RI_{\times}(G) = \prod_{(x,y)\in E(G)}\left(\frac{1}{(d(x).d(y))}\right)^{\frac{1}{2}}$$

We express this in terms of the unit graph $\Omega_U(R)$ with the adjacency conditions described in the preceding construction.

For a vertex $v\in B_A$,

$$d_{\Omega_U(R)}(v_{B_A}) = \sum_{A\in P_m, AUB=\{1,2,\dots,m\}}|B_A|$$

$$= \sum_{A\in P_m, AUB=\{1,2,\dots,m\}}\prod_{j\in B}(q_j-1)$$

The multiplicative Randic index can be rewritten as,

$$RI_{\times}(\Omega_U(R)) = \prod_{A\in P_m}\prod_{v_{B_A}\in B_A}\prod_{B\in P_m, AUB=\{1,2,\dots,m\}}\left(\frac{1}{\prod_{j\in B}(q_j-1)}\right)^{\frac{|E_{B_AB_B}|}{2}}$$

By expanding the summations using edge count formulas, we obtain,

$$RI_{\times}\big(\Omega_U(R)\big) = \prod_{A,B\in P_m, AUB=\{1,2,\ldots.,m\}} \left(\frac{1}{\prod_{i\in A}(q_i-1)\prod_{j\in B}(q_j-1)}\right)^{\frac{|E_{B_AB_B}|}{2}}$$

The multiplicative atom bond connectivity index (Kulli, 2016) is defined as

$$ABC_{\times}(G) = \prod_{(x,y)\in E(G)} \sqrt{\frac{d(x)+d(y)-2}{d(x)d(y)}}$$

For each edge (x, y) we substitute the degree formulas,

$$\sqrt{\frac{d(x)+d(y)-2}{d(x)d(y)}} = \sqrt{\frac{\prod_{i\in A}(q_i-1)+\prod_{j\in B}(q_j-1)-2}{\prod_{i\in A}(q_i-1)\times\prod_{j\in B}(q_j-1)}}$$

Summing overall partitions B_A, B_B we get,

$$ABC_{\times}(G) = \prod_{A,B\in P_m, AUB=\{1,2,\ldots,m\}} \left(\sqrt{\frac{\prod_{i\in A}(q_i-1)+\prod_{j\in B}(q_j-1)-2}{\prod_{i\in A}(q_i-1)\times\prod_{j\in B}(q_j-1)}}\right)^{|E_{B_AB_B}|}$$

RESULTS

Theorem 1

Let R be a commutative ring and $R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \ldots \times \mathbb{F}_{q_m}$, where $\mathbb{F}_{q_i}$ is a finite field of order q_i and $q_i = {p_i}^{k_i}$ The MRI and the MABCI of the unit graph $\Omega_U(R)$ are given by

$$RI_{\times}\big(\Omega_U(R)\big) = \prod_{A,B\in \mathcal{P}_m, A\cup B=\{1,2,\ldots.,m\}} \left(\frac{1}{\prod_{i\in A}(q_i-1)\prod_{j\in B}(q_j-1)}\right)^{\frac{|E_{B_AB_B}|}{2}}$$

and

$$ABC_{\times}\big(\Omega_U(R)\big) = \prod\nolimits_{A,B\in \mathcal{P}_m, A\cup B=\{1,2,\ldots.,m\}} \left(\sqrt{\frac{\prod_{i\in A}(q_i-1)+\prod_{j\in B}(q_j-1)-2}{\prod_{i\in A}(q_i-1)\prod_{j\in B}(q_j-1)}}\right)^{|E_{B_AB_B}|}$$

where $\mathcal{P}_m$ is the power set of $\{1,2,\ldots.,m\}$ including empty set,

$\prod_{i\in A}(q_i-1)$ is the total number vertices in B_A,

$|E_{B_AB_B}|$ is the total edges between partitions B_A and B_B.

The adjacency condition is defined such that $A \cup B = \{1,2,\ldots.,m\}$

ensuring that $r \oplus s \in U(R)$.

The notations used in this theorem follow from the support-based partition and H-join construction described in the Materials and Methods section. To demonstrate the relevance of the formulae, selected examples are computed below.

Calculated values of multiplicative indices of unit graphs of commutative ring

The explicit calculation of selected multiplicative indices using the main results is given in this section.

Example 2

Let $R = \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$ product of two finite fields. Then $m = 2$, $P_m = \{B_{\emptyset}, B_{\{1\}}, B_{\{2\}}, B_{\{1,2\}}\}$. For the empty support, $|B_{\emptyset}| = 1$. For single index support $B_{\{i\}}$ where only one component is nonzero, $|B_{\{i\}}| = (q_i-1)$. For two index support $B_{\{i,j\}}$ where two components are nonzero, $|B_{\{i,j\}}| = (q_i-1)(q_j-1)$. For example if we select $q_1 = 2$

and $q_2 = 2^2$, then $|B_\emptyset| = 1, |B_{\{1\}}| = 1$, $|B_{\{2\}}| = 3$, $|B_{\{1,2\}}| = 3$. This unit graph is presented in the following Figure 3.

By our main results, the MRI and MABCI for this example are as follows,

$$RI_\times(\Omega_U(R)) = \left(\frac{1}{(q_1-1)(q_2-1)}\right)^{\frac{|E(B_{\{1,2\}}B_{\{1,2\}})|+|E(B_{\{1,2\}}B_{\{1\}})|+|E(B_{\{1,2\}}B_{\{2\}})|+|E(B_{\{1\}}B_{\{2\}})|}{2}}$$

$$RI_\times(\Omega_U(R)) = \left(\frac{1}{(q_1-1)(q_2-1)}\right)^{\frac{|E(\Omega_U(R))|}{2}}$$

Substituting the values for $q_1 = 2$ and $q_2 = 2^2$, we have

$RI_\times(\Omega_U(R)) = \left(\frac{1}{(2-1)(4-1)}\right)^{\frac{12}{2}} = 0.00137.$

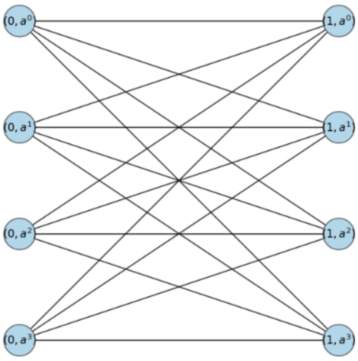

Figure 3. Unit graph associated with $\mathbf{R} = \mathbb{F}_2 \times \mathbb{F}_{2^2}$

For the $MABCI$,

$$ABC_\times(\Omega_U(R)) = \prod_{(A,B)\in P_m, A\cup B=\{1,2\}} \left(\sqrt{\frac{(q_i-1)+(q_j-1)-2}{(q_i-1)(q_j-1)}}\right)^{|E(B_A B_B)|}$$

$ABC_\times(\Omega_U(R))$

$$= \left(\sqrt{\frac{(q_1-1)+(q_2-1)-2}{(q_1-1)(q_2-1)}}\right)^{|E(B_{\{1,2\}}B_{\{1,2\}})|+|E(B_{\{1,2\}}B_{\{1\}})|+|E(B_{\{1,2\}}B_{\{2\}})|+|E(B_{\{1\}}B_{\{2\}})|}$$

$$ABC_\times(\Omega_U(R)) = \left(\sqrt{\frac{(q_1-1)+(q_2-1)-2}{(q_1-1)(q_2-1)}}\right)^{|E(\Omega_U(R))|} = 0.2963$$

Example 3

Let $R = F_{q_1} \times F_{q_2} \times F_{q_3}$, the product of three finite fields. The partition set is $P_3 = \{B_\emptyset, B_{\{1\}}, B_{\{2\}}, B_{\{3\}}, B_{\{1,2\}}, B_{\{1,3\}}, B_{\{2,3\}}, B_{\{1,2,3\}}\}$,where the cardinalities of each equivalence class are

$|B_\emptyset| = 1, |B_{\{1\}}| = q_1 - 1, \quad |B_{\{2\}}| = q_2 - 1, |B_{\{3\}}| = q_3 - 1,$

$|B_{\{1,2\}}| = (q_1-1)(q_2-1), |B_{\{1,3\}}| = (q_1-1)(q_3-1), |B_{\{2,3\}}| = (q_2-1)(q_3-1),$

$|B_{\{1,2,3\}}| = (q_1-1)(q_2-1)(q_3-1).$

The Multiplicative Randić Index for this example is given by

$$RI_{\times}\big(\Omega_U(R)\big) = \left(\frac{1}{(q_1-1)(q_2-1)(q_3-1)}\right)^{\frac{|E(\Omega_U(R))|}{2}}$$

$$RI_{\times}\big(\Omega_U(R)\big)$$

$$= \left(\frac{1}{(q_1-1)(q_2-1)(q_3-1)}\right)^{\frac{1}{2}\times\left(\frac{(q_1-1)(q_2-1)(q_3-1)\left((q_1-1)(q_2-1)(q_3-1)-1\right)}{2}\right)}$$

$$\times \left(\frac{1}{(q_1-1)(q_2-1)(q_3-1)}\right)^{\left((q_1-1)(q_2-1)(q_3-1)\times(q_2(q_3-1)+q_3(q_2-1)-4(q_1+q_2+q_3)+10)\right)}$$

$$\times \left(\frac{1}{(q_1-1)(q_2-1)(q_3-1)}\right)^{\left((q_1-1)(q_2-1)(q_3-1)\times(q_1(q_3-1)+q_3(q_1-1)-4(q_1+q_2+q_3)+10)\right)}$$

$$\times \left(\frac{1}{(q_1-1)(q_2-1)(q_3-1)}\right)^{\left((q_1-1)(q_2-1)(q_3-1)\times(q_1(q_2-1)+q_2(q_1-1)-4(q_1+q_2+q_3)+10)\right)}$$

$$\times \left(\frac{1}{(q_1-1)(q_2-1)(q_3-1)}\right)^{\left((q_1-1)(q_2-1)(q_3-1)\times 3\right)}$$

The $MABCI$ is given by

$$ABC_{\times}\big(\Omega_U(R)\big) = \left(\sqrt{\frac{(q_1-1)+(q_2-1)+(q_3-1)-2}{(q_1-1)(q_2-1)(q_3-1)}}\right)^{|E(\Omega_U(R))|}$$

$$ABC_{\times}\big(\Omega_U(R)\big) =$$

$$\left(\frac{(2(q_1-1)(q_2-1)(q_3-1)-2)}{(q_1-1)^2(q_2-1)^2(q_3-1)^2}\right)^{\frac{1}{2}\times\left(\frac{(q_1-1)(q_2-1)(q_3-1)\times\left((q_2(q_3-1)+q_3(q_2-1)-4(q_1+q_2+q_3)+10)\right)}{2}\right)}$$

$$\times\left(\frac{(2(q_1-1)(q_2-1)(q_3-1)-2)}{(q_1-1)^2(q_2-1)^2(q_3-1)^2}\right)^{\left(\frac{(q_1-1)(q_2-1)(q_3-1)\times\left((q_1(q_3-1)+q_3(q_1-1)-4(q_1+q_2+q_3)+10)\right)}{2}\right)}$$

$$\times\left(\frac{(2(q_1-1)(q_2-1)(q_3-1)-2)}{(q_1-1)^2(q_2-1)^2(q_3-1)^2}\right)^{\left(\frac{(q_1-1)(q_2-1)(q_3-1)\times\left((q_1(q_2-1)+q_2(q_1-1)-4(q_1+q_2+q_3)+10)\right)}{2}\right)}$$

$$\times\left(\frac{(2(q_1-1)(q_2-1)(q_3-1)-2)}{(q_1-1)^2(q_2-1)^2(q_3-1)^2}\right)^{\left(\frac{3(q_1-1)(q_2-1)(q_3-1)}{2}\right)}$$

$$\times\left(\frac{(2(q_1-1)(q_2-1)(q_3-1)-2)}{(q_1-1)^2(q_2-1)^2(q_3-1)^2}\right)^{\left(\frac{(q_1-1)(q_2-1)(q_3-1)\times\left((q_1-1)(q_2-1)(q_3-1)-1\right)}{4}\right)}$$

The two examples above demonstrate that, with just the values $q_1, q_2, \ldots, q_m$, it is possible to compute the Randic and atom bond connectivity multiplicative indices analytically.

DISCUSSION

The results show that the additive definition of adjacency in unit graphs leads to a partition structure different from the one usually obtained in zero-divisor graphs. In zero-divisor graphs, adjacency is governed by multiplication, whereas in the present unit graph, adjacency depends on whether the coordinate-wise sum belongs to the set of units. This difference explains why support sets and complement-type conditions become central in the computation of the degree sequences and edge partitions.

The H-join description provides an efficient way to avoid direct enumeration of all edges when the ring is a direct product of finite fields. Once the support classes are identified, the degrees and the number of edges within and between classes can be obtained combinatorially. The examples indicate that the MRI and MABCI are sensitive to the field orders and to the distribution of vertices across support classes, which makes these indices useful for distinguishing structural patterns in unit graphs.

The present work is restricted to unit graphs arising from direct products of finite fields. The same strategy may be extended to broader classes of finite commutative rings, but additional care is required when nilpotent elements or non-field components are present because the unit condition and the corresponding support partitions may change.

CONCLUSION

This paper establishes explicit combinatorial formulae for the Multiplicative Randić Index and the Multiplicative Atom-Bond Connectivity Index of unit graphs associated with direct products of finite fields. The construction is based on support-based vertex partitions and the H-join representation of the unit graph.

The main importance of the study is that it provides a repeatable method for computing these multiplicative indices without listing every edge individually. The obtained formulae and examples show that the algebraic structure of the underlying ring directly influences the degree distribution, edge partition, and resulting topological indices of the corresponding unit graph.

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AUTHORS CONTRIBUTION

Conceptualization: VR; Methodology: VR; Validation: BS; Data analysis: BS, VR; Writing-original draft: VR; Writing- review & editing: BS, VR; Supervision: BS

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CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

ETHICAL STATEMENT

The authors state that this manuscript is their original work and has not been previously published or submitted for publication elsewhere. Since the study is purely theoretical and mathematical in nature, it does not involve human participants, animals, clinical data, or biological samples. Therefore, ethical approval from an Institutional Review Board is not applicable.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are included within the article. Any further information related to the computations is available from the corresponding author upon reasonable request.

SUPPLEMENTARY INFORMATION

None

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