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Research Article

WEAKER FORMS OF COMMUTING MAPPINGS IN METRIC AND MENGER PROBABILISTIC METRIC SPACE

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ABSTRACT

This paper aims to discuss the update of the comparative study of non-commuting mappings in metric space and probabilistic metric space. This interrelationship study in weaker commuting maps helps researchers understand, analyze, and reach their research goal.

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INTRODUCTION

In 1942, Karl Menger proposed the probabilistic metric space, a generalization of metric space that uses a probabilistic distance function called a distribution function instead of the distance function used in the Frechet metric space. Menger's probabilistic metric space solves the uncertainty cases of the distance between two points in space.

Stephen Banach introduced contraction mapping in 1922, which became a metric fixed point theory source. V. M. Sehgal generalized and defined this idea in Menger space in 1966. However, this mapping is too restrictive and works only with single self-mappings. So, mathematician researchers seek more than one self-mapping. In 1968, K. Goebel worked on two self-mappings and established the coincidence theorem to derive the common fixed point theorem. Even though Machuca initially examined this issue in 1967, under some strict topological restrictions.

To establish common fixed point theorems for contractive type mappings, require a commutativity condition, a restriction on the ranges of mappings, the continuity of one or more mappings, and a contractive condition. Furthermore, obtaining a necessary version of one or more of these conditions or weakening them is the objective of all significant and common fixed-point theorems. To fulfill such a gap, Jungck introduced commuting mappings and established common fixed point theorems by using constructive procedures of the sequence of iterations in metric space in 1976. This condition is too strong. So, naturally, it needs weaker forms. In 1982, Sessa gave weakly commuting mapping and extended a variety of fixed point theorems by substituting weakly commutativity for commutativity. Thereafter less restrictive contractive mapping, compatible mapping introduced by Jungck in 1986 in metric space and its counterpart in Menger space defined by Mishra in 1991. The elegancy of this result many authors have introduced various other contractive conditions on more than one self-mapping like compatible type conditions, readers may see references (Agarwal et al., 2014; Chang, 1981; Chaudhary, 2023, 2024; Dhage, 1999; Fisher & Sessa, 1989; Hadzic & Pap, 2010; Pant, 1994, 1998, 1999; Pant et al., 2011; Singh & Pant, 1983; Sintunavarat & Kumam, 2011; Rhoades & Sessa, 1986).

This paper focuses on the comparative and interrelationship study of weaker forms of commuting mapping in metric and Menger space. It also includes variants of compatible mappings with different types.

PRELIMINARIES

Definition 2.1 A ***metric space*** is an ordered pair (X, d), where X is an abstract set and d is a mapping of $X \times X \to R$, satisfying the following axioms:

M_1: $d(p,q) = 0$ if and only if $p = q$ (Identity);
M_2: $d(p,q) \geq 0$ (Positivity);
M_3: $d(p,q) = d(q,p)$ (Symmetry);
M_4: $d(p,q) \leq d(p,r) + d(r,q)$ (Triangle inequality).
(Fréchet, 1906)

Definition 2.2 A function F: $\mathbb{R} \to \mathbb{R}^+$ is said to be a ***distribution function*** if a function is non-decreasing, left continuous with $\inf\{F(x) : x \in \mathbb{R}\} = 0$, and $\sup\{F(x) : x \in \mathbb{R}\} = 1.$
(Menger, 1942)
Heavy side function H can be taken as an example of a distribution function:

$$H(x) = \begin{cases} 0, & if\ x \leq 0 \\ 1, & if\ x > 0\ . \end{cases}$$

Definition 2.3 Let $F: X \times X \to L$ (set of all distribution functions) be a distribution function, i.e., F associates a distribution function $F(p,q)$ with every pair (p,q) of points in a non-empty set X. Then, a pair

(X, F) is said to be a ***probabilistic metric space*** (abbreviated as ***pm-space***) if the distribution function $F\,(p, q)$, also denoted by $F_{p,q}$, satisfies the following conditions:

(I) $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$,
(II) $F_{p,q}(0) = 0$ for every $p, q \in X$,
(III) $F_{p,q}(x) = F_{q,p}(x)$ for every $p, q \in X$, and
(IV) $F_{p,q}(x + y) = 1$ if and only if $F_{p,r}(x) = 1$ and $F_{r,q}(y) = 1$.

Here, $F_{p,q}(x)$ represents the value of $F_{p,q}\ at\ x \in \mathbb{R}$. *(Menger, 1942)*

Example 2.1 Let (X, d) be metric space where $X = [0, 2]$ with usual metric $d(x, y) = |x - y|$ and distribution function F defined as:

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}}, & if\ t > 0, \\ 0, & if\ t = 0. \end{cases} \quad \text{for all } x, y \in X.$$

Then, (X, F) is pm space.

Definition 2.4 A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is referred to as a ***Triangular norm*** (shortly T-norm) if it satisfies the following conditions:

T₁: $T\,(0, 0) = 0$,
T₂: $T\,(a, 1) = a$ for all $a \in [0, 1]$,
T₃: $T\,(a, b) = t\,(b, a)$ for all $a, b \in [0, 1]$,
T₄: $T\,(a, b) \le T(c, d), if\ a \le c, b \le d$ and
T₅: $T\,(t\,(a, b), c) = T\,(a, t\,(b, c))$,
where $a, b, c, d \in [0, 1]$.

Example 2.2 $T(a, b) = \max\{(a + b) - 1, 0\}$ and $T(a, b) = \min\{a, b\}$. (Hadzic & Pap, 2010)

Definition 2.5 A ***Menger space*** is a triplet (X, F, T), where X is a non-empty set, F is a function defined on $X \times X$ to the set of distribution functions and T is a t-norm such that the followings are satisfied:

(I) $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$,
(II) $F_{p,q}(0) = 0$ for every $p, q \in X$,
(III) $F_{p,q}(x) = F_{q,p}(x)$ for every $p, q \in X$, and
(IV) $F_{p,q}(t + s) \ge T(F_{p,r}(t), F_{r,q}(s))$, for every $t, s > 0\ \&\ p, q, r \in X$.

Definition 2.6 Let A and B be two self-mappings on X. Then, for some $x \in X$ is called the ***coincident point*** of A and B if $z = Ax = Bx$. z is called the point of coincidence of A and B. *(Agarwal et. al, 2014)*

Definition 2.7 Let A and B are two self-mappings on X. Then, a point $x \in X$ is said to be a ***common fixed point*** of A and B if $x = Ax = Bx$. *(Agarwal et. al, 2014)*
In 1976, Jungck defined commuting mapping in metric space as:

Definition 2.8 Let A and B are two self- mappings on X in metric space (X, d). Then, a pair (A, B) is said to be ***commuting mapping*** if and only if $ABx = BAx$ for all $x \in X$.
In 1982, Sessa weakens the commuting mapping in metric space as follows:

Definition 2.9 Let A and B are two self- mappings on X in metric space (X, d). Then, a pair (A, B) is said to be ***weakly commuting mapping*** if and only if $d(ABx, BAx) \le d(Ax, Bx)$ for all $x \in X$.
In 1983, Singh and Pant defined this mapping in probabilistic metric space as:

Definition 2.10 Two self- mappings A and B in probabilistic metric space (X, F) are said to be ***weakly commuting mapping*** if and only if $F_{ABx,BAx}(t) \ge F_{Ax,Bx}(t)$, for every $x \in X, t > 0$.
In 1986, Jungck introduced a new class of mappings, called compatible mappings in metric space:

Definition 2.11 Two mappings $A, B: X \to X$ are said to be ***compatible mappings*** in metric space (X, d) if and only if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X.
In 1991, Mishra gave the following statements for compatible mappings in Menger space:

Definition 2.12 Two mappings $A, B: X \to X$ are said to be ***compatible mappings*** in Menger space (X, F, t) if $\lim_{n\to\infty} F_{ABx_n,\ BAx_n}(t) = 1$ for all $t > 0$, whenever sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z in X.
In 1994, Pant introduced the concept of the following $R-$weakly commuting mappings in metric space and stated as;

Definition 2.13 A pair (A, B) of self-mappings of a metric space (X, d) is said to be ***$R-$weakly commuting*** if there exists some real number $\mathbb{R} > 0$ such that $d(ABx, BAx) \le Rd(Ax, Bx), for\ all\ x \in X$.

Definition 2.14 A pair (A, B) of self-mappings of a metric space $(X,\ d)$ is said to be ***point-wise R-weakly commuting*** on X, if given $x \in X$, there exists $\mathbb{R} > 0$ such that $d(fgx, gfx) \le Rd(fx, gx)$.

Remark 2.1 It is noticed from the above definition that A and B can fail to be pointwise R-weakly commuting only if there exists some $x \in X$ such that $Ax = Bx$ but $ABx \ne BAx$, i.e., only if they possess a coincidence point at which they do not commute. *(*Pant, 1994*)*

In 2007, Kohli and Vasistha extended the concepts of weak commutativity and its variants to probabilistic metric spaces:

Definition 2.15 Two self-mappings A and B of a probabilistic metric space (X, F) are said to be

(i) ***R − weakly commuting*** if there exists a positive real number ℝ such that $F_{ABx,\ BAx}(t) \geq F_{Ax,\ Bx}(\frac{t}{R})$ for each $x \in X$, and $t > 0$.

(ii) ***pointwise R − weakly commuting*** on X if given $x \in X$, there exists a positive real number ℝ such that $F_{ABx,\ BAx}(t) \geq F_{Ax,\ Bx}(\frac{t}{R})$ for all $t > 0$.

(iii) ***R − weakly commuting of type (i)***, if there exists a positive real number ℝ such that $F_{AAx,\ BAx}(t) \geq F_{Ax,\ Bx}(\frac{t}{R})$ for each $x \in X$, and $t > 0$.

(iv) ***R − weakly commuting of type (ii)***, if there exists a positive real number ℝ such that $F_{ABx,\ BBx}(t) \geq F_{Ax,\ Bx}(\frac{t}{R})$ for each $x \in X$, and $t > 0$.

(v) ***weakly commuting of type (iii),*** if there exists a positive real number ℝ such that $F_{AAx,\ BBx}(t) \geq F_{Ax,\ Bx}(\frac{t}{R})$ for each $x \in X$, and $t > 0$.

Example 2.3 Let, $X = \mathbb{R}$, and distribution function defined by $F_{x,y}(t) = \begin{cases} e^{\frac{-d(x,y}{t}} & if\ t > 0 \\ 0, & if\ t = 0. \end{cases}$ for all $x, y \in X$. Then, (X, F) is a probabilistic metric space. Let, $A, B: X \to X$ be defined by $A(x) = 2x - 1, and\ B(x) = x^2$. Then, $F_{ABx,\ BAx}(t)\ e^{\frac{-2|x-1|^2}{t}}$, and $F_{Ax,\ Bx}\left(\frac{t}{R}\right) = e^{\frac{-R|x-1|^2}{t}}$. Therefore, for $R = 2$, (A, B) is R −weakly commutings. However, (A, B) is not weakly commuting mappings since the exponential function is strictly increasing.

In 1995, Cho *et al.* introduced the non-symmetric concept of semi-compatible mappings as follows:

Definition 2.16 Two self-mappings A and B of a metric space (X, d) are said to be ***semi-compatible*** if

(i) $Ax = Bx \Rightarrow ABx = BAx;$

(ii) $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some x in $X \Rightarrow \lim_{n\to\infty} d(ABx_n, Bx) = 0.$

In 2004, Singh and Jain generalized the notion of semi-compatibility in probabilistic metric space as follows:

Definition 2.17 Two self-mappings A and B of a probabilistic metric space (X, F) are said to be ***semi-compatible*** if $\lim_{n\to\infty} F_{ABx_n,\ Bz}(t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some x in X as $n \to \infty$.

Example 2.4 Let (X, d) be a metric space where $X = [0,1]$, and the distribution function is defined by $F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & if\ t > 0 \\ 0, & if\ t = 0. \end{cases}$ for all $x, y \in X$. Then, (X, F) is pm space. Let $A, B: X \to X$ be defined by $A(x) = \begin{cases} x, & if\ 0 \leq x < \frac{1}{2} \\ 1, & if\ \frac{1}{2} \leq x \leq 1. \end{cases}$ and

$$B(x) = \begin{cases} 1 - x, & if\ 0 \leq x < \frac{1}{2} \\ 1, & if\ \frac{1}{2} \leq x \leq 1. \end{cases}$$

Taking sequence $x_n = \frac{1}{2} - \frac{1}{n}$. We obtain $Ax_n = \frac{1}{2} - \frac{1}{n}$, $Bx_n = \frac{1}{2} + \frac{1}{n}$. Then, $Ax_n \to \frac{1}{2}$, $Bx_n \to \infty, as\ n \to \infty$. Further, $ASx_n = 1$, $BAx_n = \frac{1}{2} + \frac{1}{n}$. Now, $\lim_{n\to\infty} F_{ABx_n,\ BAx_n}(t) = \lim_{n\to\infty} F_{1,\ \frac{1}{2}+\frac{1}{n}}(t) = \frac{t}{t+\frac{1}{2}} = \frac{2t}{2t+1} < 1, for\ all\ t > 0$. Hence, (A, B) is not compatible with mapping. Also, $\lim_{n\to\infty} F_{ABx_n,\ Bz}(t) = \lim_{n\to\infty} F_{1,\ 1}(t) = 1$. And it shows that (A, B) is semi-compatible. On the other hand, $\lim_{n\to\infty} F_{BAx_n,\ Az}(t) = \lim_{n\to\infty} F_{\frac{1}{2},\ 1}(t) = \frac{t}{t+\frac{1}{2}} = \frac{2t}{2t+1} < 1, for\ all\ t > 0 \Rightarrow$ pair (B, A) is not semi- compatible.

In 1997, Pant proposed the notion of non-compatible mappings in metric spaces as:

Definition 2.18 Let A and B be two self-mappings on X in metric space (X, d). Then, a pair (A, B) is said to be non-compatible mapping if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X but $\lim_{n\to\infty} d(ABx_n, BAx_n)$ is either non-zero or non-existent.

In 2010, Ali *et al.* extended this concept to pm space as follows:

Definition 2.19 Two self-mappings A and B of a probabilistic metric space (X, F) are said to be ***non-compatible mapping*** if $\lim_{n\to\infty} F_{ABx_n,\ BAx_n}(t) \neq 1$, for some $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X as $n \to \infty$.

In 1998, Jungck and Rhoades termed a pair of self-mappings to be weakly compatible in metric space and defined as:

Definition 2.20 Two self-mappings A and B of a metric space (X, d) are said to be ***weakly compatible*** if they commute at their coincidence points, that is, if $ABx = BAx$, whenever $Ax = Bx$ for some $x \in X$.

Note: it is also called coincidently commuting (Dhage, 1999), Partially commuting (Sastry et al., 2000), and compatible type(N) (Shrivastava *et al.*, 2000).

In 2005, Singh and Jain extended the notion of weakly compatible mappings in pm spaces as follows:

Definition 2.21 Two self-mappings A and B of a probabilistic metric space (X, F) are said to be ***weakly compatible*** mappings if they commute at their coincidence points, that is if $Ax = Bx$, for some $x \in X$ then $ABx = BAx$.

In 2002, Aamri and Moutawakil defined an (E.A.) property that generalized the concept of non-compatible mappings.

Definition 2.22 Let A and B be two self-mappings on X in metric space (X, d). Then, a pair (A, B) is said to satisfy ***E. A. property*** if there exists a sequence $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X.

Note: It may be observed that the (E.A.) property is equivalent to the previously known notion of tangential mappings introduced by (Sastry *et al.*, 2000).
In 2008, Kubiaczyk and Sharma extended the notion of (E.A) property to pm space as follows:

Definition 2.23 Let A and B be two self-mappings on X in probabilistic metric space (X, F). Then, a pair (A, B) is said to satisfy ***E. A. property*** if there exists two sequence $\{x_n\}$, in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X.
In 2010, Ali *et al.* defined the notion of common property (E.A) to pm spaces as follows:

Definition 2.24 Two self -mappings pair (A, B) and (S, T) are said to satisfy common property (E. A.) there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$, for some z in X.

Example 2.5 Let (X, d) be a metric space where $X = [-1,1]$, and the distribution function is defined by
$$F_{x,y}(t) = \begin{cases} e^{\frac{-d(x,y}{t}} & if\ t > 0 \\ 0, & if\ t = 0. \end{cases}$$ for all $x, y \in X$. Then, (X, F) is a probabilistic metric space.
Let, $A, B, S, T : X \to X$ be defined by $A(x) = \frac{x}{2}, and\ B(x) = \frac{-x}{2}, S(x) = \frac{x}{4}, T(x) = \frac{-x}{4}$, for all $x \in X$
Taking sequence $x_n = \frac{1}{n}$, and $y_n = \frac{-1}{n}$ in X. Then, we see that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0 \in X$, which shows that (A, B) and (S, T) are common property (E. A.).

Most common fixed point theorems for contraction mappings invariably require a compatibility condition besides the continuity of at least one mapping. In 1998, Pant noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity, and obtained a common fixed point theorem using the compatibility in metric. He also showed that in common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings.

Definition 2.25 A pair (A, B) of self-mappings of a metric space (X, d) is said to be ***reciprocal continuous*** if $\lim_{n\to\infty} ABx_n = Az$, and $\lim_{n\to\infty} BAx_n = Bz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X (Pant, 1998).

Definition 2.26 A pair (A, B) of self-mappings of a metric space (X, d) is said to be ***weakly reciprocal continuous*** if $\lim_{n\to\infty} ABx_n = Az$, **or** $\lim_{n\to\infty} BAx_n = Bz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X (Pant *et al.*, 2011).

Remark 2.2 If A and B both mappings are reciprocal continuous then they are weakly reciprocal continuous but the converse is not true (Pant *et al.*, 2011).

Definition 2.27 Two self-mappings A and B of a metric space (X, d) are called weakly uniformly contraction mappings iff $d(ABx, BBx) \leq d(Ax, Bx)$ and $d(AAx, BAx) \leq d(Ax, Bx)$ for all x in X (Pathak, 1990).

In 2008, Kumar and Pant extended the notion of reciprocal continuity to pm spaces as follows:

Definition 2.28 A pair (A, B) of self-mappings of a probabilistic metric space (X, F) is said to be ***reciprocal continuous*** if $\lim_{n\to\infty} ABx_n = Az$, and $\lim_{n\to\infty} BAx_n = Bz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$, for some z in X.
In 2011, Sintunavarat, and Kumam defined CLR property in metric space as follows:

Definition 2.29 Let A and B be two self-mappings on X in metric space (X, d). Then, a pair (A, B) is said to be a ***common limit in the range (CLR)*** of B property if there exists a sequence in $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Bx$, for some x in X.

Remark 2.3 E. A. property and ***(CLR)*** property both are for the notion of non-compatibility. These properties are well suited for studying common fixed points of strict contractive conditions, non-expansive type mapping pairs, or Lipschitz type mapping pairs in ordinary metric spaces, which are not even complete. (Agarwal *et al.*, 2014).

In 2015, Aoua and Aliouche extended this in Menger space as:

Definition 2.30 A pair of self-mappings A and B of a Menger space (X, F, t) is said to satisfy the ***common limit range property*** to the mapping B (briefly CLRs property) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} F_{Ax_n,z}(t) = F_{Bx_n,z}(t) = 1$, for some $z \in X$, and all $t > 0$.

Example 2.6 Let (X, F, t) be a Menger space with $X = [0, \infty)$ and for all $p, q \in X$ by
$F_{p,q}(t) = H(t - |p - q|), t > 0$ and $F_{p,q}(0) = 0$ where $t(x, y) = min\{x, y\}$ for all $x, y \in [0, 1]$. Define self-mappings A and B on X by $Ap = p + 3, Bp = 4p$. Let a sequence $\{x_n = 1 + \frac{1}{n}\}$, $n \in N$ in X. Since, here, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 4$, then $\lim_{n\to\infty} F_{Ax_n,\ 4}(t) = F_{Bx_n,\ 4}(t) = 1$, where $4 \in X$. Therefore, the mappings A and B satisfy the CLRs property.

In 1993, Jungck *et al.* defined compatible mappings of type (A) in metric space as:

Definition 2.31 A pair (S, T) of self-mappings of a metric space (X, d) is said to be ***compatible mappings of type (A)*** if and only if $\lim_{n\to\infty} d(SSx_n, TSx_n) = 0$, and $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some z in X.
The counterpart of compatible of type (A) in Menger Space was introduced by Cho *et al.* in 1992 as follows:

Definition 2.32 Two mappings $S, T: X \to X$ are said to be ***compatible of type (A)*** in Menger Space (X, F, t) iff $\lim_{n\to\infty} F_{STx_n,\ TTx_n}(t) = 1$ and
$\lim_{n\to\infty} F_{TSx_n,\ SSx_n}(t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.
In 1994, Pathak *et al.* introduced compatible mappings of type (P) in metric space as:

Definition 2.33 A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (P) if and only if $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some z in X.
In 2021, Chaudhary *et al.* introduced the following compatible mapping of type (P) in Menger space:

Definition 2.34 Two mappings $S, T: X \to X$ are said to be ***compatible mappings of type (P)*** in Menger space (X, F, t) iff $\lim_{n\to\infty} F_{SSx_n,\ TTx_n}(t) = 1\ \forall\, t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.
In 2023, A.K. Chaudhary introduced the following weakly compatible mapping of type (P) in Menger space:

Definition 2.35 Two mappings $S, T: X \to X$ are said to be ***Weakly Compatible Mapping of type(P)*** in Menger Space (X, F, t) iff $\lim_{n\to\infty} F_{SSx_n,\ TTx_n}(x) \geq F_{Sx_n,\ Tx_n}(x)\ \forall\, x > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Example 2.7 Let (X, d) be metric space where $X = [0, 2]$with usual metric $d(x, y) = |x - y|$ and (X, F) be PM space with

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}}, & if\ t > 0, \\ 0, & if\ t = 0. \end{cases} \quad \text{for all } x, y \in X.$$

We define S and T as:

$$S(x) = \begin{cases} 1 - x, & for\ x \in [0, 1/2) \\ 1, & for\ x \in [\frac{1}{2}, 2] \end{cases} \quad \text{and}$$

$$T(x) = \begin{cases} x, & for\ x \in [0, 1/2) \\ 1, & for\ x \in \left[\frac{1}{2}, 2\right]. \end{cases}$$

Taking sequence $\{x_n\}$ in X where $x_n = \frac{1}{2} - \frac{1}{n}$, $n \in N$.
Then, (S, T) are weakly compatible mappings of type (P) and it is neither compatible mappings of type (P) nor compatible mappings.
In 2007, Singh and Singh proposed compatible mappings of type (E) in metric space as:

Definition 2.36 A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (E) if and only if $\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} TSx_n = Sz$, and $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Tz$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some z in X.
The extension of compatible of type (E) in Menger space is as follows:

Definition 2.37 Two self-mappings $S, T: X \to X$ are said to be ***Compatible mapping of type (E)*** in Menger space (X, F, t) iff $\lim_{n\to\infty} F_{SSx_n,STx_n,Tz}(t) = 1$ and $\lim_{n\to\infty} F_{TTx_n,TSx_n,\ Sz}(t) = 1\ \forall\, t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.
The notion of compatible mappings of type (K) in metric space was introduced by Manandhar *et al.* in 2014.

Definition 2.38 A pair (S, T) of self-mappings of a metric space (X, d) is said to be compatible mappings of type (K) if and only if $\lim_{n\to\infty} d(SSx_n, Tz) = 0$, and $\lim_{n\to\infty} d(TTx_n, Sz) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some z in X.
In Menger space, it is extended by Chaudhary *et al.* in 2022 and defined as:

Definition 2.39 Two self-mappings $S, T: X \to X$ are said to be ***Compatible mapping of type (K)*** in Menger space (X, F, t) iff $\lim_{n\to\infty} F_{SSx_n,Tz}(t) = 1$ and $\lim_{n\to\infty} F_{TTx_n,\ Sz}(t) = 1\ \forall\ t > 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Example 2. 8 Let (X, d) be metric space where $X = [0,2]$ and (X, F, t) be Merger space with

$$F_{x,y}(t) = \frac{t}{t + d(x,y)} \text{ for } t > 0$$
$$= 0 \text{ for } t > 0,$$
for all $x, y \in X$, and $t > 0$.

We define A and S as:

$$S(x) = \begin{cases} 2, & for\ x \in [0,1] - \left\{\frac{1}{2}\right\} \\ 0 & \text{for } x = \frac{1}{2} \\ \frac{2-x}{2} & \text{for } x \in (1,2] \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 0, & for\ x \in [0,1] - \left\{\frac{1}{2}\right\} \\ 2 & \text{for } x = \frac{1}{2} \\ \frac{x}{2} & \text{for } x \in (1,2] \end{cases}$$

Now, taking $\{x_n\}$ in X where $x_n = 1 + \frac{1}{n}, n \in N$. Then, it is neither compatible mappings of type (A) nor compatible mappings of type (P) but (S, T) is compatible mappings of type (K).

In 2008, AI-Thapagi and Shahzad introduced the concept of occasionally weakly compatible mappings in metric space:

Definition 2.40 A pair (S, T) of self-mappings of a metric space (X, d) is said to be ***occasionally weakly compatible (OWC)*** if $STx = TSx$ for some $x \in C(S, T)$.

In the sense of Jungck and Rhodes, A pair (S, T) of self-mappings of a metric space (X, d) is said to be occasionally weakly compatible (OWC) if there exists at least one coincidence point at which S and T commute, i.e., if $ST = TS$ for some $x \in X$, then $STx = TSx$.
It is extended to Menger space by Pant *et al.* in 2011 as:

Definition 2.41 Two self-mappings $S, T: X \to X$ are said to be ***occasionally weakly compatible mapping (shortly owc)*** in Menger space (X, F, t) if there is a point x in X which is a coincidence point of S and T at which S and T commute.

MAIN RESULTS

Here, we have shown the interrelationship of metric and probabilistic metric space, and the interrelation of weaker forms of commuting mappings through examples, and charts.

Interrelationship I

Every metric space is a probabilistic metric space.

Pf: Every metric space can be shown as a probabilistic metric space if we set
$F_{p,q}(x) = H(x - d(p,q))$ for every pair of points (p, q) in the metric space.
This can be illustrated as follows:

$$F_{p,q}(x) = H(x - d(p,q))$$
$$= H(x - 0) iff\ p = q$$
$$= H(x) = 1, \quad x > 0$$ as $H(x)$ is a distribution function. This shows that $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$.

For proving $F_{p,q}(0) = 0$, consider
$F_{p,q}(0) = H(0 - d(p,q)) = 0\ as - d(p.q) < 0.$
For proving, $F_{p,q}(x) = F_{q,p}(x)$, consider

$$F_{p,q}(x) = H(x - d(p,q))$$
$$= H(x - d(q,p)) \text{ as } d(p,q) = d(q,p)$$
$$= F_{q,p}(x)$$

Finally, we consider

$$F_{p,r}(x) = H(x - d(p,r))$$
$$= H(x) = 1, \Rightarrow \mathrm{d(p,r)} = 0 \Rightarrow \mathrm{p = r}.$$
$$F_{r,q}(y) = H(y - d(r,q))$$
$$= H(y) = 1, \Rightarrow \mathrm{d(r,q)} = 0 \Rightarrow \mathrm{r = q}.$$

Therefore,

$$F_{p,q}(x + y) = H(x + y - d(p,q))$$
$$= H(x + y), \text{ for } d(p,q)) = 0, \text{ for every } \mathrm{p = q}.$$
$$= 1, \text{as } \mathrm{x > 0, y > 0 \Rightarrow x + y > 0}.$$

Thus, $F_{p,q}(x + y) = 1$ if and only if $F_{p,r}(x) = 1$ and $F_{r,q}(y) = 1$.

Interrelationship II

Every pair of commuting self-mappings is weakly commuting but the converse is not true.

(Agarwal *et al.*, 2014)

Example 2.3 Let $X = \{x, y, z\}$ and distribution function F be defined via

$$F_{x,z}(t) = F_{z,x}(t) = F_{z,y}(t) = F_{y,z}(t) = \begin{cases} 0, & if\ t \le 0, \\ \frac{1}{2}, & if\ 0 < t \le 2, \\ 1, & if\ t > 2. \end{cases}$$

And $$F_{x,y}(t) = F_{y,x}(t) = \begin{cases} 0, & if\ t \le 0, \\ \frac{1}{2}, & if\ 0 < t \le \frac{3}{2}, \\ 1, & if\ t > \frac{3}{2}. \end{cases}$$

Then, (X, F, t_{min}) is a Menger space. Let $A, B : X \to X$ be such that $A(x) = A(y) = x$, and $A(z) = y$,
$B(x) = B(y) = B(z) = x$. Then, a pair (A, B) is weekly commuting but not commuting.

Weakly commuting mappings are compatible but the converse does not hold. *(Agarwal et al, 2014)*

Example 2.4 Let $A, B: X \to X$ defined by $A(x) = x^3$, $B(x) = 2x^3$, for all x, where $X = [0, \infty)$, and d be usual metric. Then, $d(ABx, BAx) > d(Ax, Bx)$ which shows that A and B are compatible but not weakly commuting, and also not commuting mapping.

Remark 2.1 It is noticed that weak commutativity is essentially a point property, while the notion of compatibility uses the machinery of sequences (Agarwal *et al.*, 2014).

Remark 2.2 Compatibility or weak commutativity of a pair of self-mappings on a metric space depends on the choice of the metric (Agarwal *et al.,* 2014).

Commuting mappings are R- weakly commuting. (Pant, 1994)

Example 2.5 $X = [1, \infty)$, and d is the usual metric on X. Defining $A, B: X \to X$ by $Ax = 2x - 1$, $Bx = x^2$, for all $x \in X$. Here, $d(ABx, BAx) = 2(x-1)^2, and\ d(Ax, Bx) = (x-1)^2$.

So, A and B are R −weakly commuting with $R = 2$. But since $d(ABx, BAx) = 2(x-1)^2 \neq 0$,
for all $x \neq 1 \in X \Rightarrow A$ and B are not commuting.

Remark 2.3 **(**i**)**These mappings are not necessarily continuous at a fixed point.

(ii) Every R −weakly commuting pair is weakly commuting if $R = 1$.

(iii) Weak commutativity ⇒ R-weak commutativity. But R-weak commutativity ⇒
weak commutativity only when $R \leq 1$. (Pant, 1994)

Semi-compatible mappings need not be compatible mappings. (Cho *et al.*, 1995)

Example 2.7 Let $X = [2,6]$ and d be the usual metric on X. Defining $A, B: X \to X$ by
$Ax = 2\ if\ x < 3, Ax = 4\ if\ x = 3, Ax = \frac{x+21}{12}\ if\ 3 \leq x \leq 6$,
$Bx = 2, Bx = 2x\ if\ 2 < x \leq 3, Bx = \frac{2x}{3}, if\ 3 \leq x \leq 6$.
Then, A and B are semi-compatible but not compatible mapping.

Each pair of compatible self-mappings is weakly compatible, but the converse is not true.
(Jungck *et al.*, 1998)

Example 2.9 Let $X = [0, \infty)$ be metric space with usual metric. Let $A, B: X \to X$ be defined by
$A(x) = \begin{cases} x, & if\ x \in [0,1) \\ 1, & if\ x \in [1, \infty). \end{cases}$ and $B(x) = \frac{x}{1+x}, if\ x \in X$,
then a pair (A, B) is not compatible on X but commute at their coincident point $x = 0$. Indeed (A, B) is weakly compatible.

If A and B are both mappings continuous, then they are reciprocal continuous, but the converse is not true.

Example 2.11 Let $X = [2,20]$ and d be the usual metric on X. Defining $A, B: X \to X$ by
$A(x) = \begin{cases} 2, & if\ x = 2 \\ 3, & if\ x > 2. \end{cases}$, and $B(x) = \begin{cases} 2, & if\ x = 2 \\ 6, & if\ x > 2. \end{cases}$

Then, A and B are reciprocally continuous mappings but they are not continuous.

Compatible mapping and compatible mappings of type (A) are independent. *(Jungck et al, 1993)*

Example 2.13 Let $X = [2, 12]$, and d be the usual metric on X. Defining $S, T: X \to X$ as below:
$Sx = 2, if\ x = 2\ or\ x > 5, Sx = 12, if\ 2 < x \leq 5,$
$T2 = 2, if\ 7 < x \leq 20, Tx = 12, if\ 2 < x \leq$
$= \frac{x+1}{3}, if\ x > 5.$

Taking sequence $\{x_n\}$ as $x_n = 5 + \frac{1}{n}, n > 0$. Then, S and T are compatible with type (A), but neither commuting nor compatible mappings.

Example 2.14 Let $X = R$ equipped with the usual metric d. Defining self-mappings S and T as below:

$S(x) = x$, and $T(x) = \begin{cases} 0, & if\ x\ is\ an\ integer \\ 1, & if\ x\ is\ not\ an\ integer. \end{cases}$

Taking sequence $\{x_n\}$ as $x_n = 1 + \frac{1}{n+1}$. Then S and T are compatible mapping, but not compatible mapping of type (A).

If $Sz = Tz$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (P)); however, the converse may not be true. Further, if $Sz \neq Tz$ then compatible of type (E) is neither compatible nor compatible of type (A), compatible of type (P)) (Singh *et al.*, 2007).

Example 2.16 Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. We define self- maps S and T as $Sx = 1$, $Tx = 0$ for $x \in \left[0, \frac{1}{2}\right] - \left\{\frac{1}{4}\right\}, Sx = 0, Tx = 1\ for\ x = \frac{1}{4}$ and $Sx = \frac{1-x}{2}, Tx = \frac{x}{2}$ for $x \in (\frac{1}{2}, 1]$. S and T are not continuous at $x = \frac{1}{2}, \frac{1}{4}$.
Suppose that $x_n \to \frac{1}{2}, x_n > \frac{1}{2}$ for all n. Then, we have $Sx_n = (1 - x_n)/2 \to \frac{1}{4} = z$ and $Tx_n = \frac{xn}{2} \to \frac{1}{4} = z$. Also, we have $SSx_n = S(\frac{(1-x_n)}{2}) = 1 \to 1$, $STx_n = S(\frac{x_n}{2}) = 1 \to 1, T(z) = 1$ and $TTx_n = T(\frac{x_n}{2}) = 0 \to 0, TSx_n = T(\frac{(1-x_n)}{2}) = 0 \to 0, S(z) = 0$.
Therefore, (S, T) is compatible with type (E) but the pair (S, T) is neither compatible nor compatible with type (A) (compatible with type (P)).

Example 2.17 Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. We define self- maps S and T as $Sx = Tx = \frac{1}{2}$, for $x \in [0, \frac{1}{2}), Sx = Tx = \frac{2}{3}\ for\ x = \frac{1}{2}$ and $Sx = 1 - x, Tx = x$ for $x \in (\frac{1}{2}, 1]$. Consider a sequence $\{x_n\}$ in X such that $x_n \to \frac{1}{2}, x_n > \frac{1}{2}$ for all n. Then, we have

$Sx_n = (1 - x_n) \to \frac{1}{2} = z$ and $Tx_n = x_n \to \frac{1}{2} = z$. Since, $1 - x_n < \frac{1}{2}$ for all n. We have $SSx_n = S(1 - x_n) = \frac{1}{2} \to \frac{1}{2}$, $STx_n = S(x_n) = 1 - x_n \to \frac{1}{2}$, and $SSx_n = S(x_n) = x_n \to \frac{1}{2}$, $TSx_n = T(1 - x_n) = \frac{1}{2} \to \frac{1}{2}$. Also, we have $S(z) = \frac{2}{3} = T(z)$, but $ST(z) = ST\left(\frac{1}{2}\right) = S\left(\frac{2}{3}\right) = \frac{1}{3}$, $TS(z) = TS(\frac{1}{2}) = T(\frac{2}{3}) = \frac{2}{3}$. However, $\frac{1}{3} = ST(z) = 6 = TS(z) = \frac{2}{3}$, where $z = \frac{1}{2}$. Therefore, {S, T} is compatible (compatible with type (A), compatible with type (P)); but the maps are not compatible with type (E). Moreover, it has to be noted that the maps are not commuting at the coincidence point.

Weakly compatible is occasionally weakly compatible, but the converse is not true (Chaudhary, 2023).

Example 2.19 Let $\mathbb{R}$ be a usual metric space and defining two self-mappings S *and* T by $S(x) = 3x$ and $T(x) = x^2$ for all $x \in \mathbb{R}$. We see here that $Sx = Tx$ for $0,3$. And $ST0 = TS0$ but $ST3 = TS3$. So, S and T are not weakly compatible but occasionally weakly compatible.

Interrelationship III

Below, we present how non-commuting mappings are connected with or without continuity of mappings (Fig. 1).

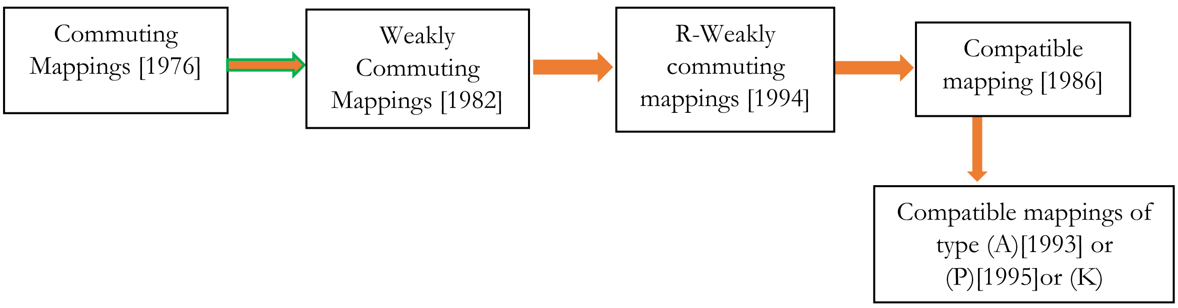

Figure 1. Non-commuting mappings

CONCLUSIONS

This article discusses comparative study of non-commuting mapping in metric and probabilistic metric space. To understand the differences, we present metric and probabilistic definitions and motivate through examples. Also, explain their interrelation and show it through a chart. Our work helps researchers for comparative studies and to solve many related open problems in this domain.

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AUTHORS CONTRIBUTIONS

DKS: Conceptualization, investigation, analyzing, writing draft; CRB: Review, editing. AKC: Conceptualization, editing, review, and corresponding.

CONFLICT OF INTEREST

The authors declare no conflict of interests.

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available from the corresponding author, upon reasonable request.

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