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Research Article

GENERALIZED KdV AND BURGER EQUATIONS AND VARIOUS VANISHING LIMITS

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ABSTRACT

In this analytical and numerical study, we look at the generalized KdV equation by letting different coefficients go to zero. Numerical study is carried out by the pseudospectral method. Our study shows that there is no difference between the behavior of the solutions in the limiting case and the solutions in the case when the corresponding coefficients are exactly zero.

Keywords: Burger equation, KdV equations, nonlinearity, vanishing viscosity

INTRODUCTION

KdV equation which arises in the study of nonlinear dispersive wave and was derived in 1895 by Korteweg and de Vries to model in shallow canal (Korteweg & Vries, 1895). The KdV equation is a nonlinear partial differential equation of third order given by

(1.1) $$u_t + u_x + 2uu_x + \delta u_{xxx} = 0$$

In this equation, $u = u(x, t)$ is a scalar function of x and t, where $x \in R$ and $t \geq 0$ and δ is a positive parameter. Physically, u represents the amplitude of the wave. The possibility of shock waves into the solution is due to nonlinear terms. Also, the term δu_{xxx} produces the dispersive broadening.

The KdV equation has numerous applications in physical sciences and engineering fields. For example, this equation is used in plasma physics, ion-acoustic solitons (Das & Sarma, 1998), geophysical fluid dynamics, long waves in shallow seas and deep oceans (Osborne, 1995; Ostrousky & Stepanyants, 1998), modeling waves in cold plasma by Kruskal. This equation has been studied by various methods such as the Tanh method (Malfliet, 1992), the sine-cosine method (Yan & Zhang, 2001), and the homogeneous balance method (Lei et al., 2002) with the appropriate initial condition (Gardner et al., 1967) . Fredner et al (cited in Gardner et al., 1967) showed the existence and uniqueness of solutions of the KdV equation and later other method such as FEM (Aksan & Özdes, 2006), FDS (Bahadir, 2005), and spectral method (Ma & Guo, 1986) were introduced. Also, artificial viscosity was used to reduce the round off error of the pseudospectral method by the author in (Rashid, 2007).

Kolabaje and Oyewande, in 2012, studied the KdV equation both analytically and numerically by using finite difference method and the Adomian decomposition method. They obtained the approximate solution. They also focused on two possible scenarios, the hyperbolic tangent initial condition and sinusoidal initial condition, and observed that the valid analytical solution is restricted to the time values close to the initial time (Kolabaje & Oyewande, 2012).

The authors (Saadi et al., 2010) presented a comparative study of the Homotopy Perturbation Method (HPM), Variation Iteration Method (VIM), and Homotopy Analysis Method (HAM) for the semi-analytical solution of the KdV equation and focused themselves to the efficiency and capability of these methods. The authors in (Karczewska et al., 2016) applied the finite element method (FEM) to obtain the numerical solution to shallow water wave which is closed with the KdV equation. This method gives a reasonable description of wave dynamics.

Among other methods, Galerkin techniques using cubic spline weight and interior functions with quintic polynomial boundary functions were used (Alexander & Morries, 1979). Also, Soliman in 2004, used the collocation method with septic splines to obtain the solution of the KdV equation (Soliman, 2004a) . The numerical solution of the KdV equation was obtained by using the variational method by the authors in (Soliman, 2006; (Inc, 2007). The modified Berstein polynomials were used for the solitons type solution of it (Zabusky, 1967) and using the method of similarity reduction for PDEs were used to develop the schemes for solving the KdV equation by the authors in (Soliman, 2000; 2004b; Soliman & Ali, 2006).

In Section 2, we explain the pseudospectral method which is used to solve the generalized KdV equation. In Section 3, we present a discussion on the specific cases, Transport equation, Burger's equation, and KdV equation, and let the values of coefficients go to zero. Section 4 concludes the paper.

SPECTRAL METHOD FOR THE GENERALIZED KDV EQUATION

The generalized KdV equation is given by

(2.1) $u_t + au_x + 2buu_x + cu_{xxx} - du_{xx} = 0,\ u(x, 0) = u_0(x)$

in the (x, t) space, where a, b, c, and d are parameters. Consider $x \in [0, 2\pi]$, $t > 0$. The interval is divided into N equal parts where N is a power of 2. The discrete transform of $u(x, t)$ is given by

$$u(\hat{x},t)=\sum_{j=0}^{N-1}u(x_j,t)e^{-\frac{2\pi i x_j k}{n}}$$; k = 0,1,2,$\cdots$, N-1

Taking the discrete Fourier transform on both sides of (2.1), we get

(2.2) $\hat{u}_t(k, t) + ika\hat{u}(k, t) + ikb\hat{u^2}(k, t) + (ik)^3c\hat{u}(k, t) - (ik)^2d\hat{u}(k, t) = 0. \hat{u}(k, 0) = \hat{u}_0(k)$

Solving (2.1) in (x, t) space is equivalent to solving (2.2) in interval $[0, 2\pi]$. To solve (2.1), we proceed as follows: Given initial function $u_0(x)$, we first find the discrete values at the N points and get a sequence $\{u_0(x_j)\}_{j=0}^{N-1}$. Then we find the discrete Fourier transform

$$u(\hat{x},t)=\sum_{j=0}^{N-1}u(x_j,t)e^{-\frac{2\pi i x_j k}{n}}$$, k= 0,1,2,$\cdots$, N-1

Taking the discrete Fourier transform on both sides of (2.1), we get

(2.2) $\hat{u}_t(k,t)+ika\hat{u}(k,t)+ikb\hat{u}^2(k,t)+(ik)^3c\,\hat{u}(k,t)-(ik)^2d\,\hat{u}(k,t)=0;\hat{u}(k,0)=\hat{u}_0(k)$

Solving (2.1) in (x,t) space is equivalent to solving (2.2) in the interval [0,2π]. To solve (2.1), we proceed as follows: Given initial function $u_0(x)$, we first find the discrete values at N points and get a sequence $\{u_0(x_j)\}_{j=0}^{N-1}$. Then we find the discrete Fourier transform $\{\hat{u}_0(k)\}_{j=0}^{N-1}$ and $\{\hat{u}_0^2(x_j)\}_{j=0}^{N-1}$ which are given by the formula

$$u_0(\hat{k},t)=\sum_{j=0}^{N-1}u(x_j,t)e^{-\frac{2\pi i x_j k}{n}}$$, k= 0,1,2,$\cdots$, N-1

and

$$u_0^2(\hat{k},t)=\sum_{j=0}^{N-1}u^2(x_j,t)e^{-\frac{2\pi i x_j k}{n}}$$, k= 0,1,2,$\cdots$, N-1.

The Fast Fourier Transform (FFT) is used to fasten process. After finding the values of $\{\hat{u}_0(k)\}_{j=0}^{N-1}$ and $\{\hat{u}_0^2(k)\}_{j=0}^{N-1}$ from FFT, (2.2) can be solved numerically by using Euler method. As for the Euler method, $\hat{u}_t(k,t)$ in the equation (2.2), we have $\frac{\hat{u}(k,h)-\hat{u}(k,0)}{h}\approx -ika\hat{u}(k,0)-ikb\hat{u}^2(k,0)+ik^3c\hat{u}(k,0)-k^2d\hat{u}(k,0)$

$$\hat{u}(k,h)\approx\hat{u}(k,0)+h[-ika\hat{u}(k,0)-ikb\hat{u}^2(k,0)+ik^3c\hat{u}(k,0)-k^2d\hat{u}(k,0)]$$

$$\hat{u}(k,h)\approx\hat{u}(k)+h[-ika\hat{u}(k)-ikb\hat{u}^2(k)+ik^3c\hat{u}(k)-k^2d\hat{u}(k)]$$

Additionally, we can also use the implicit Euler method. In this case, the formula from t = jh to t = (j+1)h is given by

$$\frac{\hat{u}(k,(j+1)h)-\hat{u}(k,jh)}{h}\approx -ika\hat{u}(k,jh)-ikb\hat{u}^2(k,jh)+ik^3c\hat{u}(k,jh)-k^2d\hat{u}(k,jh)$$

Which is due to the implicit Euler formula. After further simplification, we have

$$\hat{u}(k,(j+1)h)\approx\frac{\hat{u}(k)-ikh[a\hat{u}(k,jh)+b\hat{u}^2(k,jh)]}{1-h[ik^3c-k^2d]}$$

NUMERICAL RESULTS

The general wave equation is given by

$$u_t + au_x + 2buu_x + cu_{xxx} - du_{xx} = 0,\ u(x,0) = f(x), \tag{3.1}$$

where a, b, c, d are parameters. If $a = b = c = d = 1$, the equation (3.1) becomes the KdV equation. Our main study focuses on the behavior of the solution of the equation (3.1) as the parameters a, b, c, d tend to zero in various ways.

Transport Equation. With $b = c = d = 0$, the equation (3.1) becomes

$$u_t + au_x = 0,\ \ u(x,0) = f(x), \tag{3.2}$$

which is the transport equation. Here we have created a different set of values of a and observed the scenario as the value of a tends to zero. The graphs of the solution for the different values of the parameter are presented in the following Figures [1], [2], [3], [4], [5], [6].

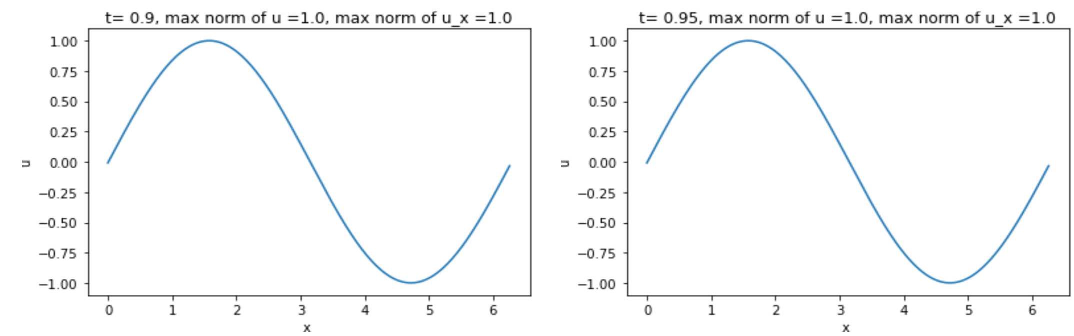

Figure 1. Plot of transport equation with a = 0.01.

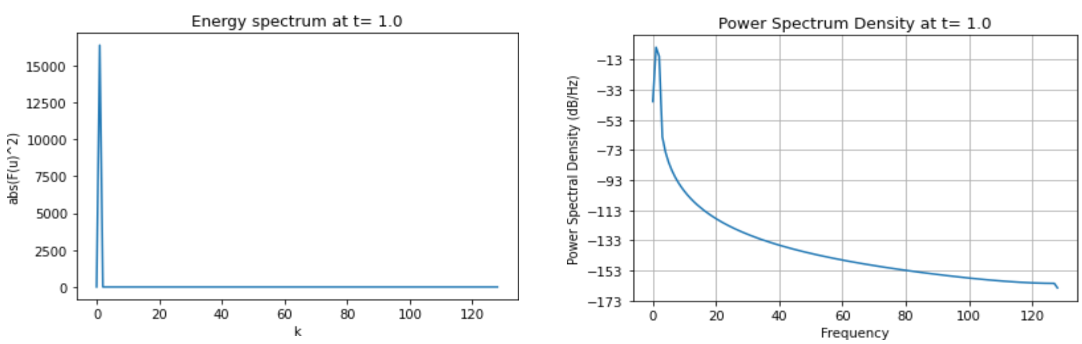

Figure 2. Plot of energy and power spectrum.

Burger Equation and Viscous Burger Equation. When $a = 0,\ b = 1,\ c = 0$, the equation (3.1) takes the form

$$u_t + 2uu_x - du_{xx} = 0,\ u(x,0) = f(x) \tag{3.3}$$

This equation is called the viscous Burger equation and if $d = 0$, then the equation (3.1) is called the inviscid Burger equation. We get the different data sets as values of d is varied and we observe the nonlinearity in the solution as the value of d tends to zero. Some of the graphs of the solutions are presented in the following Figures [7], [8], [9], [10], [11], [12], [13], [14].

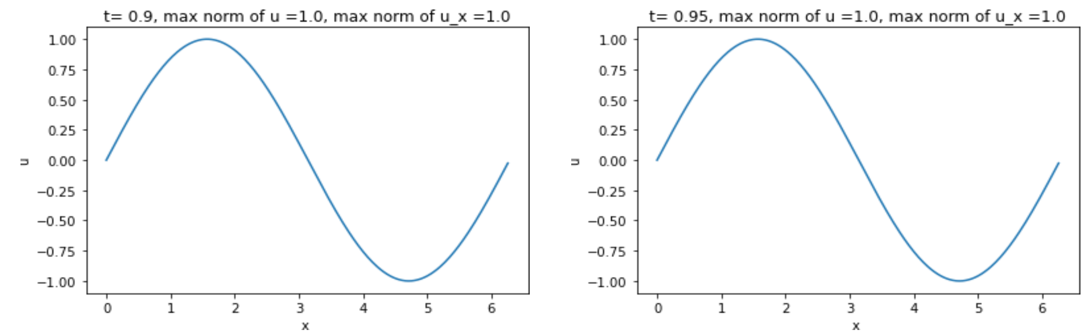

Figure 3. Plot of transport equation with a = 0.001.

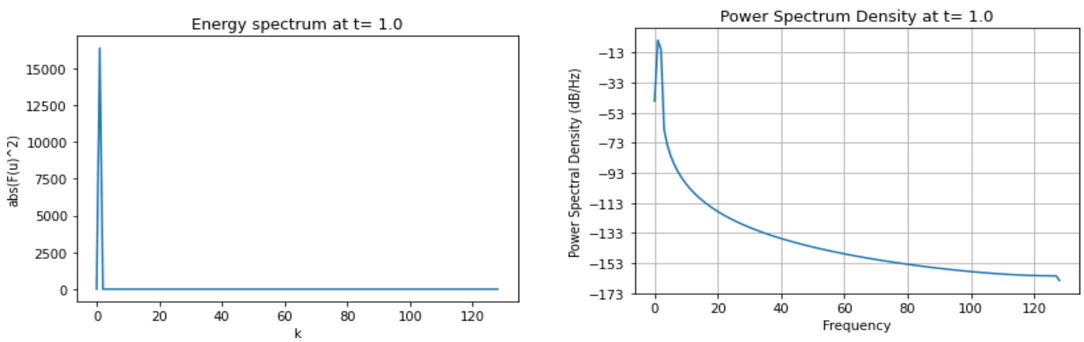

Figure 4. Plot of energy and power spectrum.

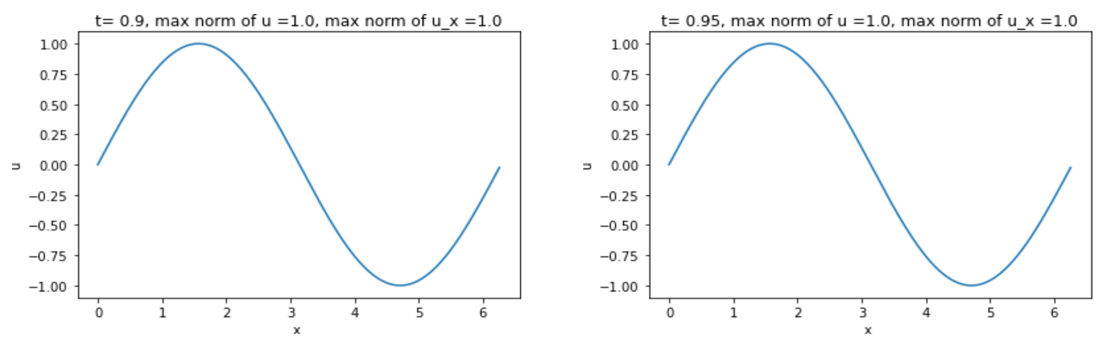

Figure 5. Plot of transport equation with a = 0.

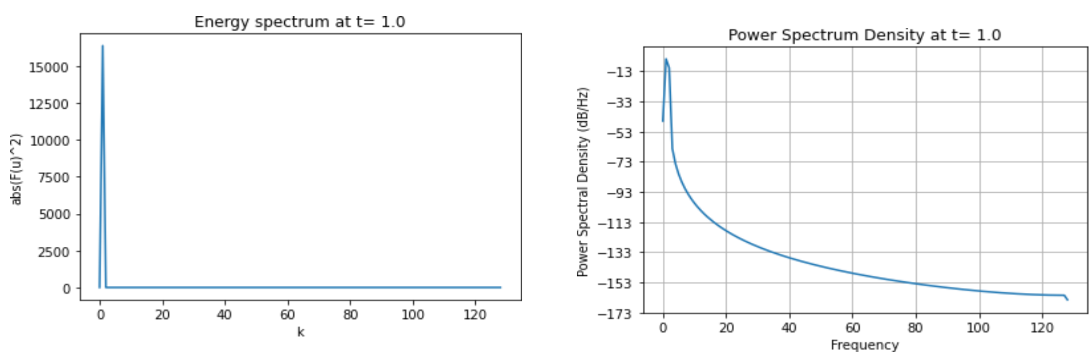

Figure 6. Plot of energy and power spectrum.

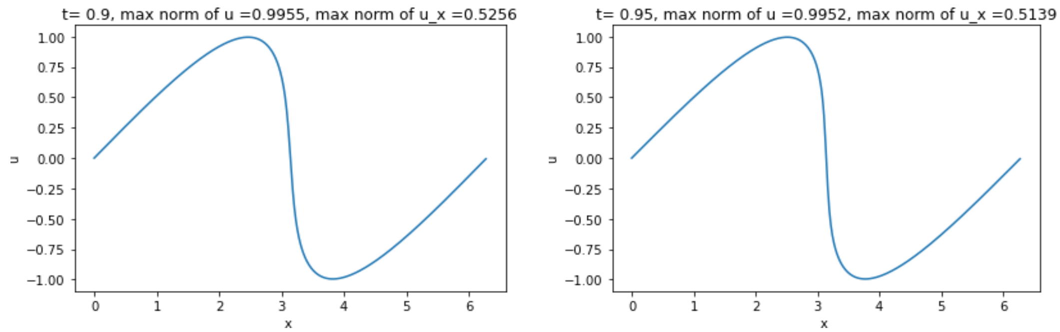

Figure 7. Plot of Burger equation b = 0.5, d = 0.005.

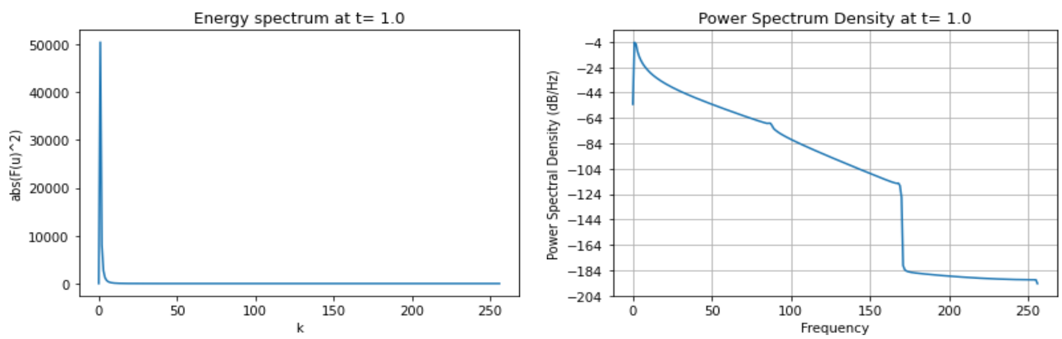

Figure 8. Plot of energy and power spectrum b = 0.5, d = 0.005.

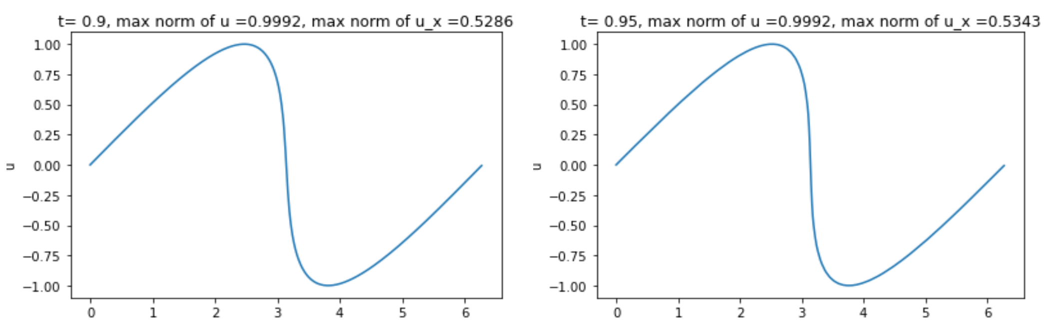

Figure 9. Plot of Burger equation b = 0.5, d = 0.0009.

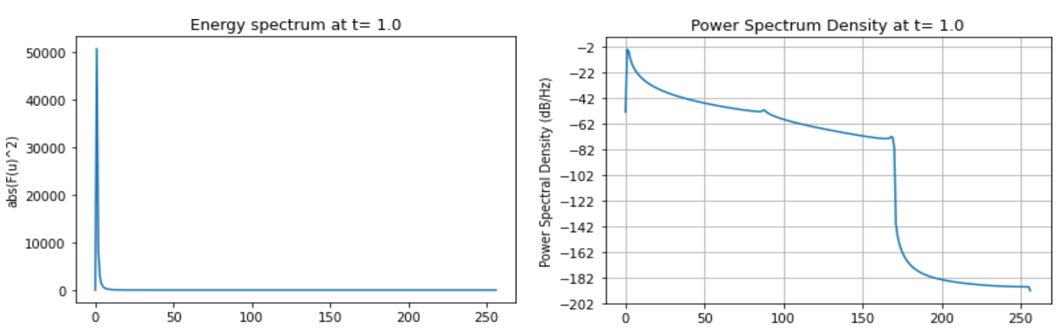

Figure 10. Plot of energy and power spectrum b = 0.5, d = 0.0009.

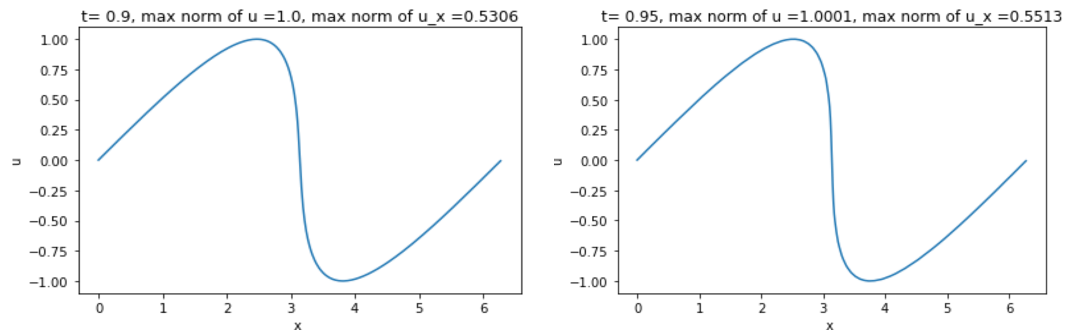

Figure 11. Plot of Burger equation b = 0.5, d = 0.0001.

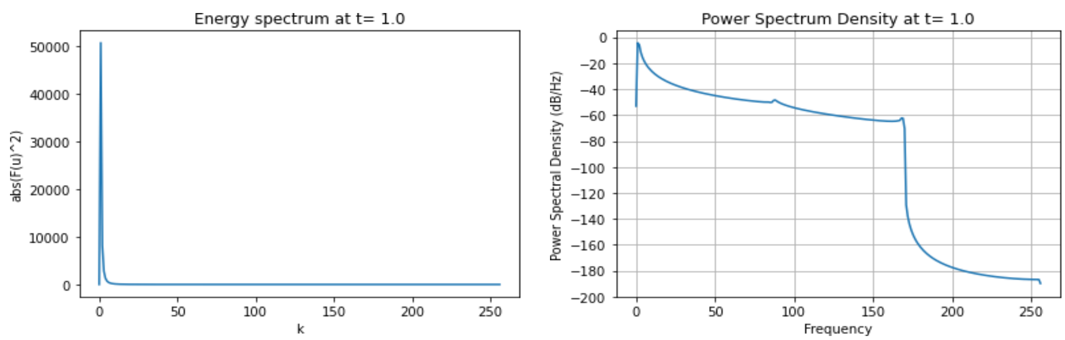

Figure 12. Plot of energy and power spectrum b = 0.5,d = 0.0001.

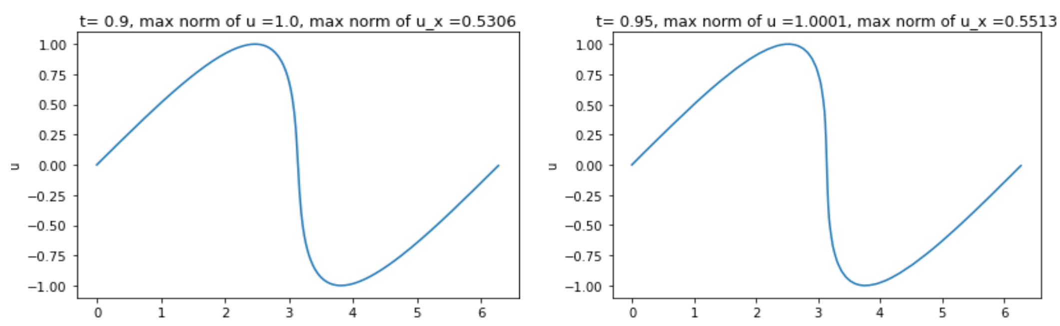

Figure 13. Plot of Burger equation b = 0.5, d = 0.

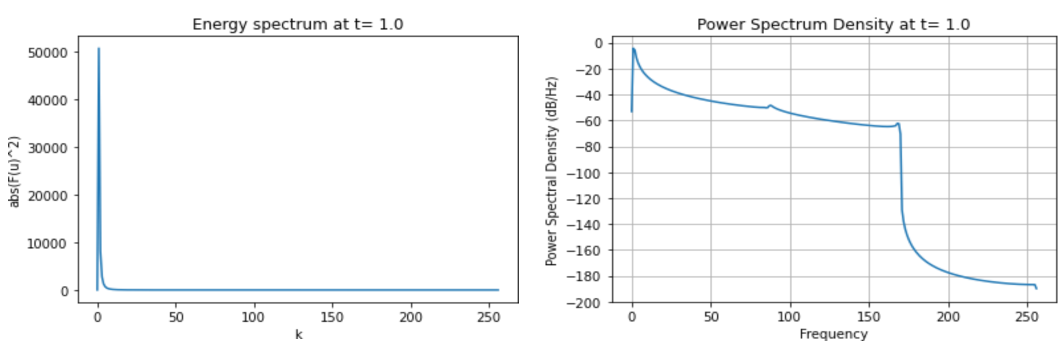

Figure 14. Plot of energy and power spectrum b = 0.5, d = 0.

The Linearized KdV Equation. With $b = d = 0$, the equation (3.1) takes the form

$$u_t + au_x + cu_{xxx} = 0, u(x, 0) = f(x), \tag{3.4}$$

which is known as the linearized Korteweg de Vries (KdV) equation. We obtain different data sets for the various values of d and observe the nonlinearity in the solution as the value of d tends to zero. Some of the graphs of the solutions are presented in Figures [15], [16], [17], [18], [19], [20].

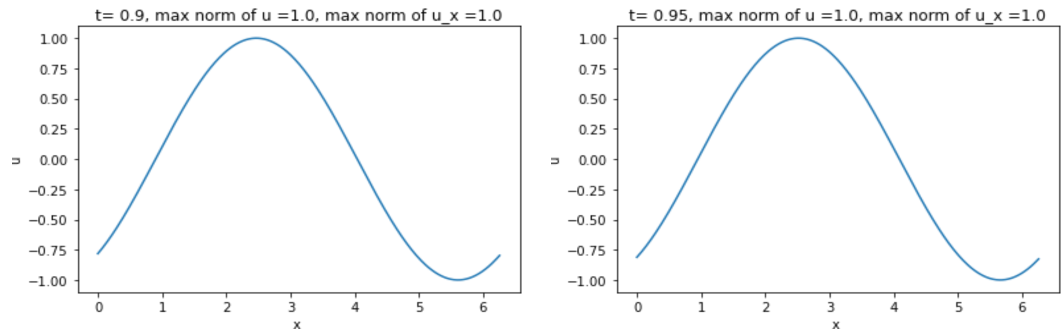

Figure 15. Plot of KdV equation with a = 1, c = 0.003.

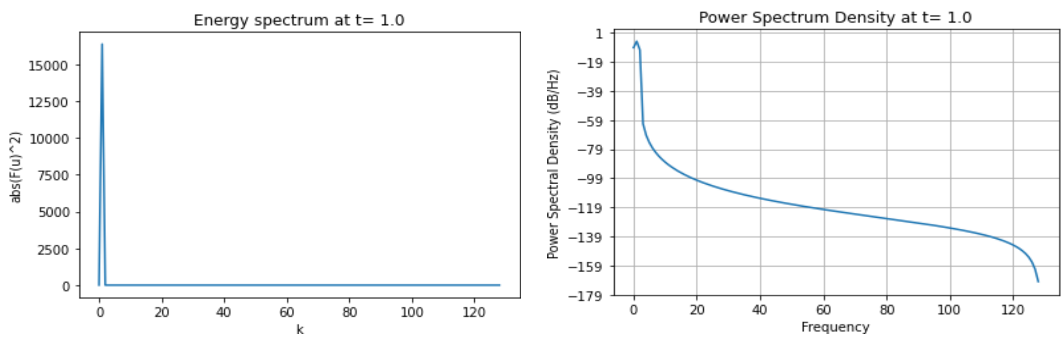

Figure 16. Plot of energy and power spectrum, a = 1, c = 0.003.

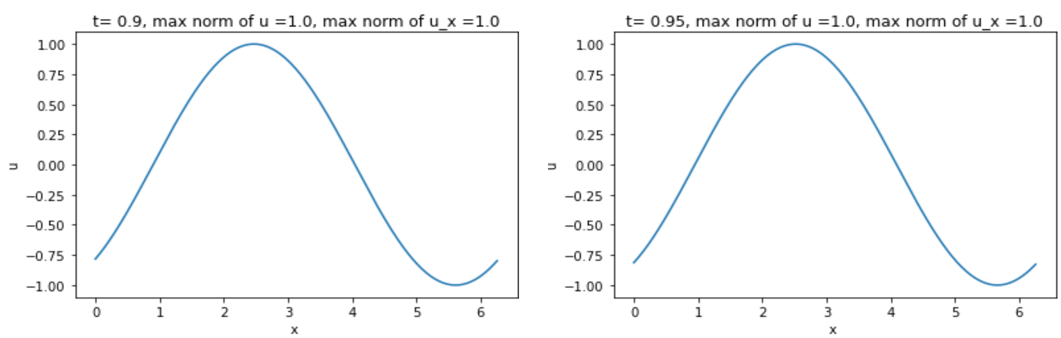

Figure 17. Plot of KdV equation with a = 1, c = 0.001.

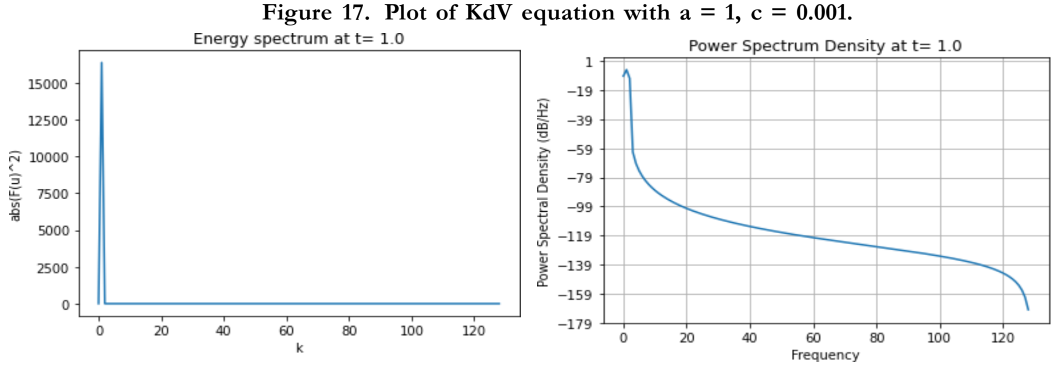

Figure 18. Plot of energy and power spectrum, a=1, c=0.001.

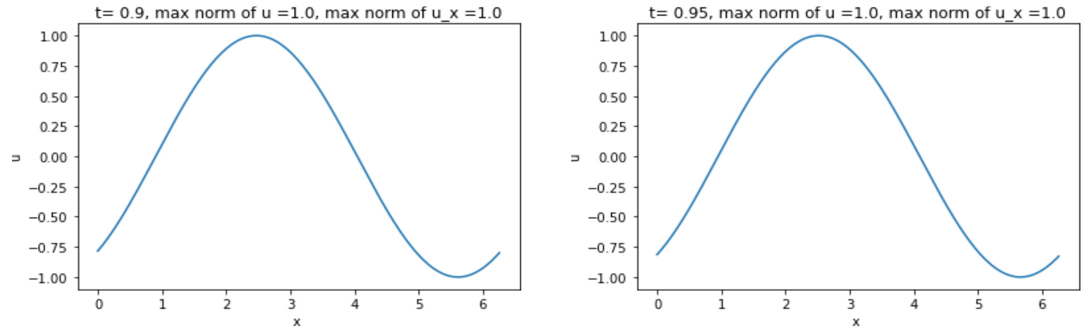

Figure 19. Plot of KdV equation with a = 1, c = 0.

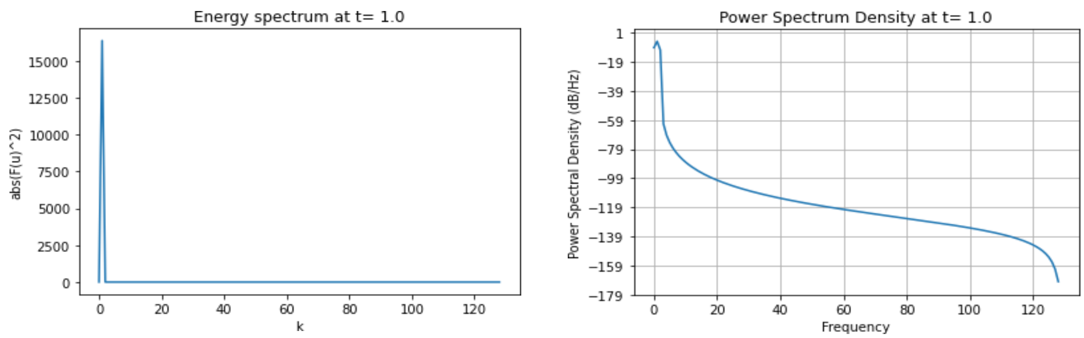

Figure 20. Plot of energy and power spectrum, a = 1, c = 0.

CONCLUSIONS

The generalized KdV equation is studied thoroughly. Then we chose the particular values of constants so as to reduce the equation to Transport, Burger's and KdV equations. We used the pseudospectral method to observe the nature of solution in two different situations when the value of the parameter is zero and another in the sense of limit that the value of parameter is zero. We found that both cases show a similar phenomenon and structure of the solution.

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AUTHOR CONTRIBUTIONS

All authors have equal contributions.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

Upon reasonable request, the data that supports this study will be made available by the authors.

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