Journal of Institute of Science and Technology, 27(1), 125-133 (2022)
ISSN: 2467-9062 (print), e-ISSN: 2467-9240
https://doi.org/10.3126/jist.v27i1.46695

Research Article

MODIFIED INVERSE NHE DISTRIBUTION: PROPERTIES AND APPLICATION

Arun Kumar Chaudhary[1*], Laxmi Prasad Sapkota[2], Vijay Kumar[3]
[1]Department of Management Science, Nepal Commerce Campus, Tribhuvan University, Nepal
[2]Department of Statistics, Tribhuvan Multiple Campus, Tribhuvan University, Palpa, Nepal
[3]Department of Mathematics and Statistics, DDU Gorakhpur University, Gorakhpur, India
**Corresponding Author: akchaudhary1@yahoo.com*
*(**Received**: January 03, 2022; **Revised**: May 20, 2022; **Accepted**: June 30, 2022)*

ABSTRACT

We have defined a modified form of the inverse NHE distribution in this study. The suggested distribution's hazard function might be shaped like a constant or increasing. Some of the suggested distribution's statistical properties are all clearly derived. Maximum likelihood, Least squares, and Cramer-Von-Mises methods are applied to determine the novel distribution's parameters. For estimators, the asymptotic confidence intervals have been determined as well. The fit of the new distribution is evaluated using an actual data set. We've found that the suggested distribution has proven to be effective in modeling real data set.

Keywords: Hazard function, inverse Nadarajah Haghighi distribution, Reliability function

INTRODUCTION

Almost all probability and applied statistics literatures show that probability models are frequently utilized in the research of failure-time analysis in numerous domains of statistics, engineering and biological disciplines. Existing models don't always yield a better fit when it comes to modeling reliability data. As a result, the majority of academics are drawn to altering traditional distributions and exploring their applicability and adaptability. These novel modified distributions, which are created by adding extra shape parameter(s) to the baseline model, usually provide good fit when compared to standard classical distributions.

In the recent couple of decades, it has been discovered that the exponential distribution is used as the basis for creating a novel family of distributions. A number of researchers made reforms to the exponential distribution. Distribution of exponential extension (EE) (Kumar, 2010), distribution of generalized inverted exponential (Abouammoh and Alshingiti ,2009) and gamma EE by (Ristic and Balakrishnan, 2012), are some of the modified exponential distributions. Lemonte (2013) introduced a novel exponential-type distribution, and its hazard function might be constant, upside-down bathtub, decreasing, bathtub-shaped, or increasing. A novel expansion of the exponential model was presented by (Gomez *et al.*, 2014). Rasekhi *et al.* (2017) created the modified exponential distribution by merging the distribution of extended exponential with the distribution of generalized exponential (Gupta & Kundu, 2001).The four-parameter (c,d,λ,θ) CDF of a modified exponential distribution with $x>0$ is

$$H(x)=1-\left\{1-\frac{cd}{c+d}ln\left[1-\left[1-e^{-\lambda x}\right]^{\theta}\right]\right\}\left\{1-\left[1-e^{-\lambda x}\right]^{\theta}\right\}^{c} \quad (1)$$

In this article, we have created the new distribution by modifying the inverse NHE, which is the inversion of exponential distribution's extension (Nadarajah & Haghighi, 2011) and is also known as NHE (Chaudhary & Kumar, 2020a) distribution. Using this NHE distribution the inverse NHE was defined by (Tahir et al., 2018) with CDF and PDF having scale parameter (β) and shape parameter (θ) with $\{x>0,(\theta,\beta)>0\}$ are

$$G(x)=e^{\{1-(1+\beta/x)^{\theta}\}} \quad (2)$$

$$g(x)=\frac{\beta\theta}{x^{2}}(1+\beta/x)^{\theta-1}\exp\left\{1-(1+\beta/x)^{\theta}\right\} \quad (3)$$

It is possible to simulate positive real data sets using this distribution, and for numerous values of α, the HRF can be formed as a decreasing or upside-down bathtub. Hence, we select this inverse NHE distribution for this study. By using similar approach, we have generated the new distribution as used by (Lai *et al.*, 2003). They had used the Weibull model as the base distribution to define the modified Weibull distribution. The Weibull distribution's CDF with $x>0,(\alpha,\theta>0)$ is

$$G(x)=1-\exp(-\alpha x^{\theta})\ . \quad (4)$$

By adding one parameter λ to (4), the modified Weibull distribution has been defined by (Lai *et al.*,2003) with $(\alpha,\theta,\lambda>0), x>0$, having CDF as

$$G'(x)=1-\mathrm{e}^{-\alpha x^{\theta}e^{\lambda x}}. \qquad (5)$$

Khan (2014) has been modified the inverse Rayleigh distribution and called it as the modified inverse Rayleigh distribution having CDF with

$$G(x)=\exp\left[-\theta/x-\beta/x^2\right].\ x>0,(\theta,\beta>0) \quad (6)$$

To model positive data sets, Khan (2015) defined a modified beta Weibull probability distribution with five parameters that is ideal. The upside down bathtub hazard rate function can be used in this distribution. By extending the ordinary Rayleigh and exponential distributions, Iriarte, *et al.* (2018) announced distribution of the modified slashed-Rayleigh. Gillariose *et al.* (2020) has defined the distribution of Marshall-Olkin modified Lindley. Many writers have also used the NHE distribution to create flexible models, such as the Burr-X Nadarajah Haghighi distribution, which was established by (Elsayed & Yousof, 2019). The Poisson inverse NHE distribution was defined by (Chaudhary & Kumar, 2020b) using the inverse NHE as the base model.

The study's main goal is to provide a more adaptable model for the inverse NHE distribution by adding a single additional parameter described by (Tahir *et al.*, 2018). The modified inverse NHE distribution's statistical and distributional features are explored, as well as its applicability. The following are the remaining sections of the intended study. The modified inverse NHE distribution and its different statistical and distributional features are defined in second section. CVM, LSE, and MLE are applied for calculating the parameters of the suggested model in third section. In section four a real data set is considered to understand how the proposed model may be used to investigate the potentiality. The estimated values of the parameters and fit statistics such as log-likelihood, AIC, BIC, and CAIC criteria are calculated. It is discovered that the suggested distribution outperforms than some well-known distributions. Final section includes some concluding remarks.

The modified inverse NHE (MINH) distribution

Assume X is a positive random variable having three parameters $(\alpha,\beta,\lambda)>0$ & $x>0$ with MINH distribution if it's CDF is

$$F(x)=\exp\left\{1-\left(1+\frac{\lambda}{x}.e^{-\beta x}\right)^{\alpha}\right\}. \qquad (7)$$

The associated PDF of (7) is

$$f(x)=(1+\beta x)\alpha\lambda x^{-2}e^{-\beta x}\left(1+\lambda x^{-1}e^{-\beta x}\right)^{\alpha-1}T_1$$
$$where\ T_1=\exp\left\{1-\left(1+\lambda x^{-1}e^{-\beta x}\right)^{\alpha}\right\} \qquad (8)$$

Survival function of MINH distribution

$$S(x;\beta,\alpha,\lambda)=1-\exp\left[1-\left(1+\mathrm{e}^{-\beta x}\frac{\lambda}{x}\right)^{\alpha}\right] \qquad (9)$$

MINH distribution's failure rate function (HRF).

The HRF is,

$$h(x)=\frac{\lambda\alpha}{x^2}(1+\beta x)e^{-\beta x}\left(1+\frac{\lambda}{x}\exp(-\beta x)\right)^{\alpha-1}T_2$$
$$where\ T_2=\left\{\left[\exp\left\{1-\left(1+\frac{\lambda}{x}\exp(-\beta x)\right)^{\alpha}\right\}\right]^{-1}-1\right\} \qquad (10)$$

The reverse hazard function of MINH distribution

The reverse failure rate function is,

$$R_{hf}=\lambda\alpha(1+\beta x)x^{-2}\left(\lambda x^{-1}\exp(-\beta x)+1\right)^{\alpha-1}\mathrm{e}^{-\beta x} \qquad (11)$$

Cumulative hazard function (CHF)

The CHF of $MINH(\alpha,\ \beta,\ \lambda)$ model defined by

$$H(x)=-\log\left[1-\exp\left\{1-\left(\lambda x^{-1}\mathrm{e}^{-\beta x}+1\right)^{\alpha}\right\}\right] \qquad (12)$$

Quantile function:

Assume T is a positive r.v. with $F_T(t)$ as its distribution function. Let $\{p\in[0,1]\}$, then T's p^th^-quantile can be defined as

$$Q_T(p)=F_T^{-1}(p)$$
$$1-\{1-\log(p)\}^{1/\alpha}+\frac{\lambda}{x}\mathrm{e}^{-\beta x}=0 \qquad (13)$$

Random Deviation Generation

The MINH's random deviation generation with [$0<u<1$] can be obtained as the solution of (14),

$$1-\{1-\ln(u)\}^{1/\alpha}+\frac{\lambda}{x}\exp(-\beta x)=0 \qquad (14)$$

The MINH distribution's kurtosis and skewness

Coefficient of skewness using quartiles is,

$$S_k(Bowley)=\frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},\ \text{and}$$

Moors (1988) calculated the coefficient of kurtosis using octiles as

$$K_u(Moors) = \frac{Q(0.375) - Q(0.125) + Q(0.875) - Q(0.625)}{Q(6/8) - Q(2/8)}$$

With various parameter values, Fig. 1 presents PDF and HRF charts for $MINH(\alpha,\ \beta,\ \lambda)$

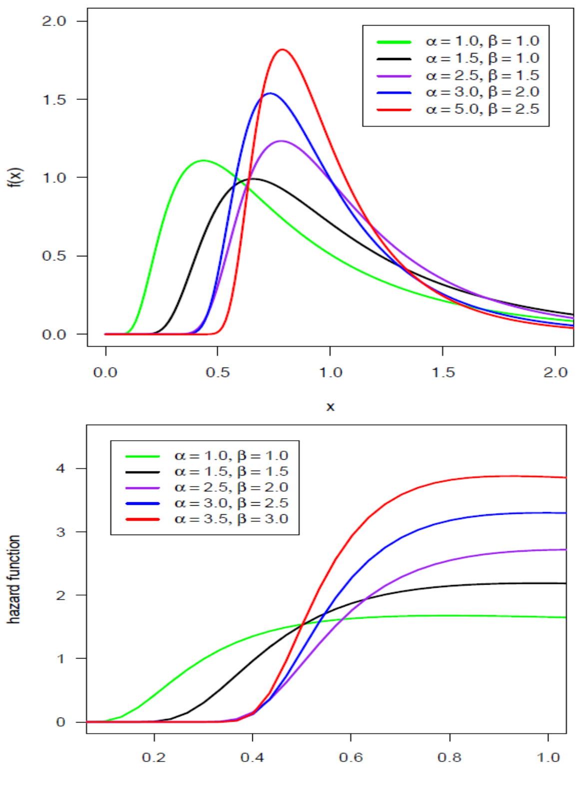


Figure 1. For various values of λ and β and for fixed α, charts of the PDF (upper section) and the hazard function (lower section).

Some useful expansion of MINH distribution

Here, to expand the CDF and PDF of the MINH model, we've used the following series expansions.

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{k} x^k$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The CDF of MINH distribution with $x > 0$ defined in (7) is

$$F(x) = \exp\left[1 - \left(1 + \lambda x^{-1} \mathrm{e}^{-\beta x}\right)^{\alpha}\right]$$

$$= \sum_{i=0}^{\infty} \frac{\left[1 - \left(\lambda x^{-1} e^{-\beta x} + 1\right)^{\alpha}\right]^i}{i!}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j}{i!} \binom{i}{j} \left(1 + \lambda x^{-1} e^{-\beta x}\right)^{\alpha j}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j}{i!} \binom{i}{j} \sum_{k=0}^{n} \binom{n}{k} \left(\lambda x^{-1} e^{-\beta x}\right)^k,$$

where $n = \alpha j$

$$\left[\because (1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k, n > o\right]$$

$$F(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \lambda^k x^{-k} e^{-\beta k x} \binom{i}{j} \frac{(-1)^j}{i!} \quad (15)$$

Differentiating (15) w. r. t. x we get PDF as,

$$f(x) = F'(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^j}{i!} \binom{i}{j} \binom{n}{k} \lambda^k T_3$$

$$\text{where } T_3 = \left[-k x^{-k} e^{-\beta k x} \left(\beta + x^{-1}\right)\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{j+1}}{i!} k \binom{i}{j} \binom{n}{k} \lambda^k x^{-k} e^{-\beta k x} \left(\beta + x^{-1}\right)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} x^{-k} e^{-\beta k x} \left(\beta + x^{-1}\right) \quad (16)$$

$$\text{where, } Z_{ijk} = \frac{(-1)^{j+1}}{i!} k \binom{i}{j} \binom{n}{k} \lambda^k$$

Moments of MINH Distribution

About the origin, the MINH distribution's r[th] moment is

$$\mu_r' = \int_0^{\infty} x^r f(x)\, dx$$

$$= \int_0^{\infty} x^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} x^{-k} e^{-\beta k x} \left(\beta + x^{-1}\right) dx$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} \int_0^{\infty} x^{r-k-1} e^{-\beta k x} dx + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} \int_0^{\infty} x^{r-k} e^{-\beta k x} \beta\, dx$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} \beta \frac{(r-k+1)^{1/2}}{(\beta k)^{r-k+1}} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} \frac{(r-k)^{1/2}}{(\beta k)^{r-k}} \quad (17)$$

where $\int_0^{\infty} x^n e^{-ax} dx = \frac{\lceil n+1}{a^{n+1}}$ is a standard gamma integral.

MINH distribution's moment generating function (mgf):

The mgf is

$$M_x(t) = E\left(e^{tx}\right) = \sum_{l=0}^{\infty} \frac{t^l}{l!} \mu_r'$$

$$= \sum_{l=0}^{\infty} \frac{t^l}{l!} \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} \beta \frac{\lceil r-k+1}{(\beta k)^{r-k+1}} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk} \frac{\lceil r-k}{(\beta k)^{r-k}}\right]$$

Mean Residual Life (MRL) Function:
MINH's MRL function is:

$$\mu(x)=E(X-t/X>t)=\frac{\int_t^{\infty} x\,f(x)dx}{1-F(x)}-x$$

$$=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk}\beta\frac{\Gamma(2-k,\beta kt)}{(\beta k)^{2-k}}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk}\frac{\Gamma(1-k,\beta kt)}{(\beta k)^{1-k}}}{1-F(x)}-x$$

Mean past life time
Random variable X's mean past life time is

$$\mu^{*}(x)=E(x-X/X\le x)=x-\frac{\int_0^{x} t\,f(t)dt}{F(x)}$$

$$=x-\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk}\frac{\gamma(1-k,\beta kt)}{(\beta k)^{1-k}}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z_{ijk}\beta\frac{\gamma(2-k,\beta kt)}{(\beta k)^{2-k}}}{1-F(x)}$$

where $\gamma(a,b)$ is lower incomplete gamma function.

Order Statistics for MINH distribution
For $X_1,...,X_n$ iid random variables from CDF F(x), let $X_{k:n}$ signify the k^th order statistic and $f_{k:n}$ denote k^th order statistic's PDF and defined by

$$f_{k:n}(x)=T_{nk}.f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k}$$

$$=T_{nk}\,f(x)\sum_{j=1}^{n-k}\binom{n-k}{j}\left[F(x)\right]^{j+k-1}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n} Z^{*}_{ijk}x^{-k}e^{-\beta kx}\left(\beta+\frac{1}{x}\right)\times\sum_{l=1}^{n-k}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{n}\psi_{ijkl}x^{-k}e^{-\beta kx}\right]^{l+k-1}$$

Where $T_{nk}=\frac{n!}{(n-k)!(k-1)!}$,

$Z^{*}_{ijk}=Z_{ijk}\times T_{nk}$ and $\psi_{ijkl}=\binom{n-k}{l}\frac{(-1)^{j}}{i!}\binom{i}{j}\binom{n}{k}\lambda^{k}$

Estimation Methods
MLE method
The MLE approach is the most often used method for estimating a model's parameter. If a random sample drawn from $MINH(\alpha,\beta,\lambda)$ be $x_1,...,x_n$, and $L(\alpha,\beta,\lambda)$ be the likelihood function with $x>0$ and then, it is given by

$$L(\phi;x_1,x_2...x_n)=f(x_1,x_2,...x_n/\phi)=\prod_{i=1}^{n} f(x_i/\phi)$$

$$L(\alpha,\beta,\lambda)=(\alpha\lambda)^{n}\prod_{i=1}^{n}x_i^{-2}(1+\beta x_i)e^{-\beta x_i}\left(1+\lambda x_i^{-1}e^{-\beta x_i}\right)^{\alpha-1}T_4$$

$$where\;\; T_4=\exp\left\{1-\left(1+\lambda x_i^{-1}e^{-\beta x_i}\right)^{\alpha}\right\}$$

The density of log-likelihood is now is

$$\ell=n\ln(\alpha\lambda)-2\ln x_i+(\alpha-1)\sum_{i=1}^{n}\ln\left(1+\frac{\lambda}{x_i}e^{-\beta x_i}\right)+T_5$$

$$where,T_5=\sum_{i=1}^{n}\ln(1+\beta x_i)+n-\sum_{i=1}^{n}\left(1+\frac{\lambda}{x_i}e^{-\beta x_i}\right)^{\alpha}-\beta\sum_{i=1}^{n}x_i \quad (18)$$

Differentiating (18) w.r. to $\alpha,\beta,and\ \lambda$, we get,

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1+\lambda x_i^{-1}\exp(-\beta x_i)\right)+\sum_{i=1}^{n}\log\left(1+\lambda x_i^{-1}e^{-\beta x_i}\right)T_6$$

$$whereT_6=\left\{1-\left(1+\lambda x_i^{-1}\exp(-\beta x_i)\right)^{\alpha}\right\}$$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{n}\left[x_i^{-1}e^{-\beta x_i}\left\{(\alpha-1)\left(1+\lambda x_i^{-1}e^{-\beta x_i}\right)^{-1}-\alpha\left(1+\lambda x_i^{-1}e^{-\beta x_i}\right)^{\alpha-1}\right\}\right]$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\left\{x_i(1+\beta x_i)^{-1}\right\}-T_7-\sum_{i=1}^{n}x_i$$

$$whereT_7=\sum_{i=1}^{n}\left[\left\{(\alpha-1)\left(1+\lambda x_i^{-1}e^{-\beta x_i}\right)^{-1}-\alpha\left(1+\lambda x_i^{-1}\exp(-\beta x_i)\right)^{\alpha-1}\right\}\left(\beta\lambda x_i^{-1}e^{-\beta x_i}\right)\right]$$

Set the previous 3 equations to zero and solve for $\alpha,\beta,and\ \lambda$ all at the same time, we acquire the ML estimates $\hat{\alpha},\ \hat{\beta}\ and\ \hat{\lambda}$ of $\alpha,\beta,and\ \lambda$. The ML estimates of $\alpha,\beta,and\ \lambda$ can be determined by utilizing software such as Matlab, R, Mathematica, and others to maximize (18). For hypothesis testing and the confidence interval estimate of $\alpha,\beta,and\ \lambda$, the observed information matrix (OIM) must be calculated. The OIM for three parameters are calculated in the following way:

$$M=\begin{bmatrix}M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{bmatrix}$$

Where

$$M_{11}=\frac{\partial^2 l}{\partial\alpha^2},M_{12}=\frac{\partial^2 l}{\partial\alpha\partial\beta},\ M_{13}=\frac{\partial^2 l}{\partial\alpha\partial\lambda}$$

$$M_{21}=\frac{\partial^2 l}{\partial\beta\partial\alpha},\ M_{22}=\frac{\partial^2 l}{\partial\beta^2},\ M_{23}=\frac{\partial^2 l}{\partial\beta\partial\lambda}$$

$$M_{31}=\frac{\partial^2 l}{\partial\lambda\partial\alpha},\ M_{32}=\frac{\partial^2 l}{\partial\beta\partial\lambda},\ M_{33}=\frac{\partial^2 l}{\partial\lambda^2}$$

If $\Omega=(\beta,\alpha,\lambda)$ and the associated MLE is $\{\hat{\Omega}=(\hat{\beta},\hat{\alpha},\hat{\lambda})\}$, then $\{\hat{\Omega}-\Omega\}\rightarrow N_3\left[0,\left(M(\Omega)\right)^{-1}\right]$ where Fisher's information matrix is $M(\Omega)$. The variance-covariance matrix is derived from OIM using the Newton-Raphson technique to optimize the probability which is defined through

$$\left[M(\Omega)\right]^{-1} = \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\beta}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\lambda}) & \text{cov}(\hat{\beta},\hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} \quad (19)$$

As a result of MLEs' asymptotic normality, nearly (1-ϑ) × 100 percent confidence intervals for three parameters with a standard normal variate $Z_{\vartheta/2}$ are calculated by

$\{\hat{\alpha} \pm Z_{\vartheta/2} \times S.E.(\hat{\alpha})\}$, $\{Z_{\vartheta/2} \times S.E.(\hat{\beta}) \pm \hat{\beta}\}$ and,

$\{Z_{\vartheta/2} \times S.E.(\hat{\lambda}) \pm \hat{\lambda}\}$

LSE method

Swain *et al.* (1988) developed weighted and simple LS estimators for calculating the Beta distribution's parameters. Minimizing (20) with regard to β, λ and α, and yields the LS estimators of β, λ and α for MINH distribution.

$$M(X;\alpha,\beta,\lambda) = \sum_{i=1}^{n}\left[F(X_i) - \frac{i}{n+1}\right]^2 \quad (20)$$

Consider $F(X_i)$, which represent the distribution function of the ordered random variables $X_{(1)} < \ldots < X_{(n)}$ and $\{X_1,\ldots,X_n\}$ represents a random sample of size n drawn from a distribution function. Then LS estimators $\{(\hat{\beta},\hat{\lambda}\ and\ \hat{\alpha})\}$ for are found by minimizing (20) with regard to $\alpha,\beta, and\ \lambda$.

$$M(X;\alpha,\beta,\lambda) = \sum_{i=1}^{n}\left[\exp\left\{1-\left(1+\lambda x_i^{-1} e^{-\beta x_i}\right)^{\alpha}\right\} - \frac{i}{n+1}\right]^2 \quad (21)$$

Differentiating (21) with regard to $\alpha,\beta, and\ \lambda$ .we have,

$$\frac{\partial M}{\partial \beta} = 2\alpha\lambda \sum_{i=1}^{n}\left[1 - e^{[1-J(x_i)^{\alpha}]} - \frac{i}{n+1}\right] e^{-\beta x_i}\, e^{[1-J(x_i)^{\alpha}]}\, J(x_i)^{\alpha-1}$$

$$\frac{\partial M}{\partial \lambda} = 2\alpha \sum_{i=1}^{n} x_i^{-1} e^{-\beta x_i} J(x_i)^{\alpha-1} e^{[1-J(x_i)^{\alpha}]}\left[1 - e^{[1-J(x_i)^{\alpha}]} - \frac{i}{n+1}\right]$$

$$\frac{\partial M}{\partial \alpha} = 2\sum_{i=1}^{n} \ln\{J(x_i)\} J(x_i)^{\alpha} e^{[1-J(x_i)^{\alpha}]}\left[1 - e^{[1-J(x_i)^{\alpha}]} - \frac{i}{n+1}\right]$$

where $J(x_i) = 1 + \frac{\lambda}{x_i} e^{-\beta x_i}$

Minimizing K w. r. t. λ, β and α and yields the weighted LS estimators.

$$K(X;\alpha,\lambda,\beta) = \sum_{i=1}^{n} w_i \left[F.(X_{(i)}) - \frac{i.}{n+1}\right]^2$$

Here, w_i represents the weights which are denoted by $w_i = \left(Var(X_{(i)})\right)^{-1} = (n+2)(n+1)^2\{i(n-i+1)\}^{-1}$

As a result, by minimizing (22) w.r.t. to { λ, β and α }, weighted LS estimators of { λ, β and α } can be produced.

$$B(X;\lambda,\alpha,\beta) = \sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{(n-i+1)i}\left[\exp\left\{1-\left(1+\frac{\lambda}{x_i}e^{-\beta x_i}\right)^{\alpha}\right\} - \frac{i}{n+1}\right]^2 \quad (22)$$

CVME method

By minimizing the function (23), it is possible to obtain the CVM estimators of { λ, β and α }.

$$T(X) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n} \mid \alpha,\beta,\lambda) - \frac{2i-1}{2n}\right]^2$$

$$= \sum_{i=1}^{n}\left[e^{\{1-(1+\lambda x_i^{-1} e^{-\beta x_i})^{\alpha}\}} - \frac{2i-1}{2n}\right]^2 + \frac{1}{12n} \quad (23)$$

Differentiating (23) w.r.t. to $\alpha,\beta, and\ \lambda$, we get,

$$\frac{\partial T}{\partial \alpha} = 2\sum_{i=1}^{n}\left[1 - e^{\{1-J(x_i)^{\alpha}\}} - \frac{2i-1}{2n}\right] J(x_i)^{\alpha} e^{\{1-J(x_i)^{\alpha}\}} \ln\{J(x_i)\}$$

$$\frac{\partial T}{\partial \beta} = 2\alpha\lambda \sum_{i=1}^{n} J(x_i)^{\alpha-1}\left[1 - e^{\{1-J(x_i)^{\alpha}\}} - \frac{2i-1}{2n}\right] e^{-\beta x_i} e^{\{1-J(x_i)^{\alpha}\}}$$

$$\frac{\partial T}{\partial \lambda} = 2\alpha \sum_{i=1}^{n} x_i^{-1} e^{-\beta x_i}\left[1 - e^{\{1-J(x_i)^{\alpha}\}} - \frac{2i-1}{2n}\right] e^{\{1-J(x_i)^{\alpha}\}} J(x_i)^{\alpha-1}$$

where $1 + \frac{\lambda}{x_i} e^{-\beta x_i} = J(x_i)$

The CVM estimators are obtained by solving $\frac{\partial T}{\partial \alpha} = 0,\ \frac{\partial T}{\partial \beta} = 0\ and\ \frac{\partial T}{\partial \lambda} = 0$ simultaneously.

Applications

We consider a real data set described by (Ghitany *et al.*, 2008) that reflects waiting times (in minutes) for 100 bank clients to demonstrate the MINH distribution's adaptability.

[38.5, 33.1, 31.6, 27.0,0.8, 0.8, 1.3, 1.5, 3.3, 3.5, 3.6, 4.0, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 7.7, 8.0, 8.2, 8.6, 8.6, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 8.6, 8.8, 8.8, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 8.9, 8.9, 9.5, 9.6, 9.7, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 6.2, 6.2, 6.2, 9.8, 13.7, 13.9, 14.1, 18.4, 21.3, 21.4, 18.9, 19.0, 19.9, 15.4, 15.4, 17.3, 18.2, 17.3, 18.1, 20.6, 21.9, 23.0]

We've approximated the MLEs of the MINH distribution using the optim function in R - software described by (R Core Team, 2021) & (Ming Hui, 2019) by optimizing (18). l = -317.0699 is the calculated Log-Likelihood value. Table 1 shows the MLEs for alpha, beta and lambda, and together with their 95 percent asymptotic confidence interval (ACI) and standard errors (S.E.).

Table 1. S.E. and MLE, $\alpha, \beta, and\ \lambda$ of MINH

(Parameters)	MLE	SE	95% ACI
α	0.48581	0.04153	(0.4044, 0.5672)
β	0.10987	0.01603	(0.0784, 0.1413)
λ	37.51293	6.47677	(24.8185, 50.2074)

Fig. 2 shows that the ML estimates for $\alpha, \beta, and\ \lambda$ are derived individually.

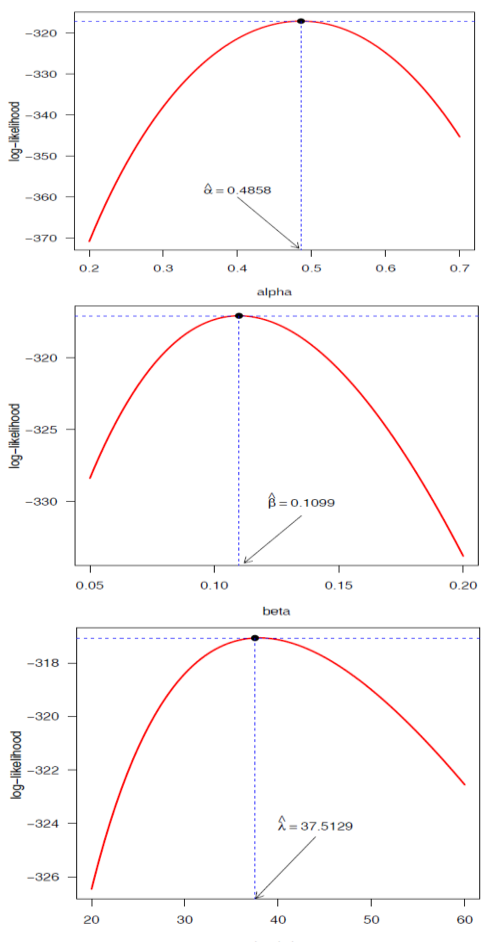


Figure 2. Graphs of Profile log-likelihood function $\alpha, \beta, and\ \lambda$

The(Q-Q) chart and (P-P) chart are shown in Fig. 3, and the MINH distribution fits the data quite well.

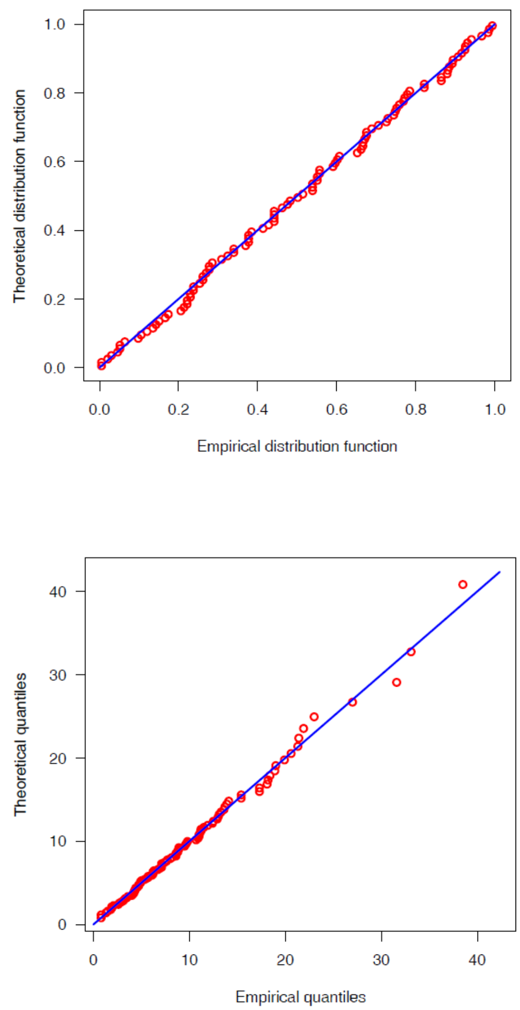
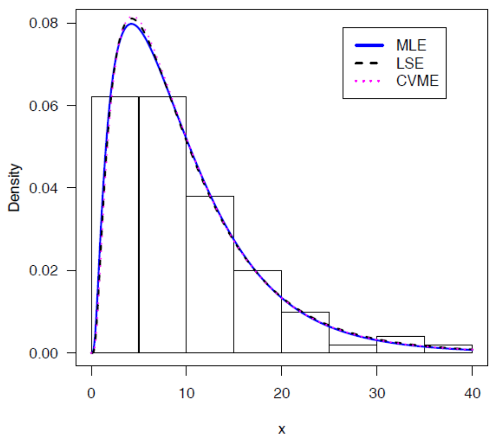


Figure 3. The (P-P) chart (right section) and (Q-Q) chart (left section) of the MINH distribution

Table 2 shows the estimated parameters values of suggested distribution using the LSE, MLE, and CVE techniques, as well as the associated AIC, negative log-likelihood, and KS statistics with p-values.

Table 2. Log-likelihood, AIC and KS statistics for estimates of LSE, MLE, and CVE techniques

Estimation Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	LL	AIC	*KS(p-value)*
MLE	0.4858	0.1099	37.5129	-317.0699	640.1398	0.0457(0.9850)
LSE	0.5264	0.1014	31.1105	-317.2991	640.5982	0.0400(0.9972)
CVE	0.5317	0.1040	31.2640	-317.3453	640.6907	0.0363(0.9994)



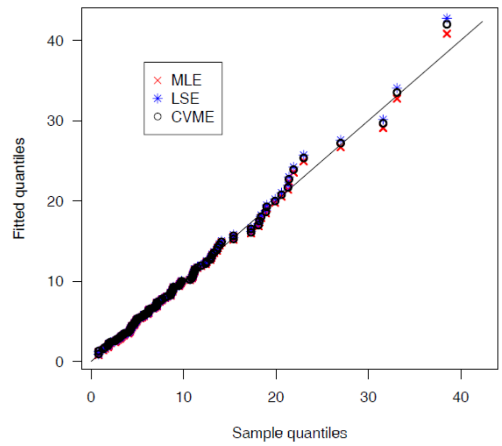


Figure 4. MLE, LSE, and CVM estimation methods' density function and histogram of fitted distributions (left section) and Q-Q plot (right section)

In this segment of fig. 4, we've shown how the MINH distribution can be useful for a real dataset. We have looked at five models to compare the suggested model's goodness of fit to that of other rival models, including the distribution of Exponential Extension (EE): NHE by (Nadarajah & Haghighi, 2011), distribution of Exponentiated Power Lindley (EPL) by (Ashour & Eltehiwy, 2015), distribution of Marshall-Olkin Extended Exponential (MOEE) by (Marshall & Olkin, 1997), distribution of Power Lindley (PL) by (Ghitany *et al.*, 2013) and Generalized Rayleigh (GR) distribution by (Kundu & Raqab, 2005).

We use some renowned goodness-of-fit statistics like log-likelihood (-LL), HQIC, AIC, BIC, and CAIC to compare these models and verify the quality of the fits of the suggested model and display in Table 3. The model having lowest goodness-of-fit statistics is the best at fitting the data.

Table 3. Log-likelihood (LL), BIC, AIC, CAIC and HQIC

	LL	AIC	BIC	CAIC	HQIC
MINH	-317.0699	640.1398	647.9553	640.3898	643.3029
EPL	-317.1008	640.2016	648.0171	640.4516	643.3646
PL	-318.3186	640.6372	645.8475	640.7609	642.7459
MOEE	-320.7120	645.4241	650.6344	645.5453	647.5328
GR	-321.5182	647.0364	652.2467	647.1601	649.1451
NHE	-323.4487	650.8973	656.1077	651.0185	653.0060

Here AIC = Akaike information criterion, HQIC = Hannan-Quinn information criterion, BIC=Bayesian information criterion, CAIC = Corrected Akaike information criterion. In Fig. 5, MINH distribution and a few additional distributions' empirical and estimated distribution functions, as well as density function and the histogram of fitted distributions, are illustrated.

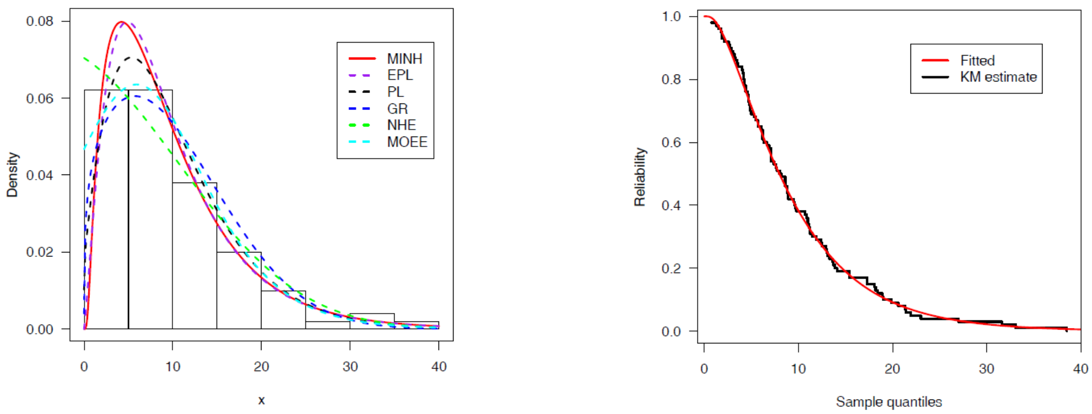


Figure 5. Fitted distributions' histogram and density function (left section) & Empirical distribution function with estimated distribution function (right section)

Table 4 shows that the values of Anderson-Darling (AD), CVM and Kolmogorov-Smniorov (KS) statistics to evaluate the MINH distribution's goodness-of-fit to other competing distributions. Because both the MINH distribution and Power Lindley (PL) distribution have the largest p-value and the lowest test statistic value, it may accomplish that both MINH and PL distributions have a far better fit and consistency than the other competing distributions.

Table 4. The p-values associated with goodness-of-fit statistics

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
MINH	0.0457(0.9850)	0.0197(0.9974)	0.1535(0.9983)
EPL	0.0375(0.9989)	0.0178(0.9987)	0.1280(0.9996)
PL	0.0520(0.9498)	0.0458(0.9025)	0.3028(0.9359)
MOEE	0.0596(0.8690)	0.0760(0.7164)	0.6351(0.6150)
GR	0.0945(0.3337)	0.2043(0.2595)	1.0911(0.3126)
NHE	0.1069(0.2028)	0.2096(0.2499)	1.5539(0.1642)

CONCLUSIONS

The modified inverse NHE distribution is investigated in this paper. The proposed model's distributional and statistical properties have been described. The hazard function can have a wide range of monotone failure rates, increasing, constant, and the MINH distribution's PDF curve can be unimodal and positively skewed. The suggested distribution fits the real dataset far better, as evidenced by the P-P and Q-Q charts. We have used a real data set to examine three popular estimation procedures: MLE, LSE, and CVM and found that ML estimates outperform then LSE and CVM estimates. An asymptotic confidence interval for MLEs has also been created. The application demonstrates that both the MINH distribution and PL distribution r outinely outperform rival distributions in terms of fit and flexibility. In the fields of probability theory and applied statistic, this model is expected to be a viable alternative.

AUTHOR CONTRIBUTIONS

AKC: formulated the concept, conducted the research, manuscript writing; LPS: data analysis, manuscript writing; VK: developed the distribution, manuscript writing.

CONFLICT OF INTERESTS

The authors declare no conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

REFERENCES

Ashour, S.K., & Eltehiwy, M.A. (2015). Exponentiated power Lindley distribution. *Journal of advanced research*, *6*(6), 895-905.

Chaudhary, A.K., & Kumar, V. (2020a). The Logistic NHE Distribution with Properties and Applications. *International Journal for Research in Applied Science & Engineering Technology, 8*(12), 591-603.

Chaudhary, A.K., & Kumar, V. (2020b). Poisson Inverse NHE Distribution. *International Journal of Science and Research, 9*(12), 1603-1610.

Elsayed, H.A.H., & Yousof, H.M. (2019). The Burr X Nadarajah Haghighi distribution: Statistical properties and application to the exceedances of flood peaks data. *Journal of Mathematics and Statistics*, *15*, 146-157.

Ghitany, M.E.,Atieh, B., & Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics and Computers in Simulation, 78*, 493-506.

Ghitany, M.E., Al-Mutairi, D.K., & Aboukhamseen, S.M., (2013). Estimation of the reliability of a stress-strength system from power Lindley distributions, *Communications in Statistics - Simulation and Computation*, *78*, 493-506.

Gillariose, J., Tomy, L., Jamal, F., & Chesneau, C. (2020). The Marshall-Olkin modified Lindley distribution: properties and applications. *Journal of Reliability and Statistical Studies, 33*(1), 177-198.

Gomez, Y.M., Bolfarine, H., & Gomez, H.W. (2014). A new extension of the exponential distribution. *Revista Colombiana de Estadistica, 37*(1), 25-34.

Gupta, R.D., & Kundu, D. (2001). Exponentiated exponential family; an alternative to gamma and Weibull. *Biometrical Journal, 43*, 117-130.

Iriarte, Y.A., Castillo, N.O., Bolfarine, H., & Gómez, H.W. (2018). Modified slashed-Rayleigh distribution. *Communications in Statistics-Theory and Methods*, *47*(13), 3220-3233.

Khan, M.N. (2015). The modified beta Weibull distribution. *Hacettepe Journal of Mathematics and Statistics*, *44*(6), 1553-1568.

Khan, M.S. (2014). Modified inverse Rayleigh distribution. *International Journal of Computer Applications*, *87*(13), 28-33.

Kumar, V. (2010). Bayesian analysis of exponential extension model. *Journal of National Academy of Mathematics*, *24*, 109-128.

Kundu, D., & Raqab, M.Z. (2005). Generalized Rayleigh Distribution: Different Methods of Estimation, *Computational Statistics and Data Analysis*, 49, 187-200.

Lai, C.D., Xie, M., & Murthy, D.N. . (2003). A modified Weibull distribution. *IEEE Transactions on reliability*, *52*(1), 33-37.

Lemonte, A.J. (2013). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. *Computational Statistics & Data Analysis*, *62*, 149-170.

Marshall, A.W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika, 84*(3), 641–652.

Ming Hui, E.G. (2019). *Learn R for applied statistics*. Springer, New York.

Moors, J. (1988). A quantile alternative for kurtosis. *The Statistician, 37*, 25-32.

Nadarajah, S., & Haghighi, F. (2011). An extension of the exponential distribution. *Statistics, 45*(6), 543-558.

R Core Team (2021). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

Rasekhi, M., Alizadeh, M., Altun, E., Hamedani, G.G., Afify, A.Z., & Ahmad, M. (2017). The modified exponential distribution with applications. *Pakistan Journal of Statistics*, *33*(5), 383-398.

Tahir, M.H., Cordeiro, G.M., Ali, S., Dey, S., & Manzoor, A. (2018). The inverted Nadarajah–Haghighi distribution: estimation methods and applications. *Journal of Statistical Computation and Simulation*, *88*(14), 2775-2798.