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Research Article

STATISTICAL PROPERTIES AND APPLICATIONS OF EXPONENTIATED INVERSE POWER CAUCHY DISTRIBUTION

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ABSTRACT

In this article, we have introduced the new distribution named exponentiated inverse power Cauchy distribution, which presents more flexibility in modeling a real lifetime dataset. The proposed distribution is analytically appealing and easy to work with and can be used efficiently to analyze the real data sets. Its probability density function can include various shapes according to the value of the parameters. Different explicit expressions for its quantile, survival, hazard and generating function, density function of the order statistics, cumulative hazard function, and failure rate function are provided. The model's parameters are estimated by using the maximum likelihood estimation method, and we also obtained the observed information matrix. We have also constructed the asymptotic confidence intervals for the estimated parameters of the proposed distribution. We have illustrated the goodness-of-fit test and the application of the purposed distribution empirically through a real-life data set. All the computations are performed in R software (version 4.1.1). It is observed that the proposed distribution gets at least similar or a better fit than some selected distributions taken for comparison.

Keywords: Cauchy distribution, MLE, order statistics, power Cauchy distribution

INTRODUCTION

In the last decades, several families of probability models have been purposed. New distributions are often created from a modification of a baseline random variable Y by; linear transformation, inverse transformation (e.g. inverse Lindley, inverse Exponential models), power transformation (e.g. Weibull is achieved from the exponential), log transformation (e.g. log gamma, log-normal, log-logistic), non-linear transformation (e.g. log-logistic from logistic), T-X family of distribution is presented by (Alzaatreh *et al.*, 2013), the compounding of some discrete and important lifetime distributions (e.g. the Poisson-X family distribution) (Tahir *et al.*, 2016). A given linear combination or mixture of baseline models usually defines a class of probability distribution having baseline as a special case. The Cauchy distribution is symmetric, uni-modal, and bell-shaped having a much heavier tail as compared to the Gaussian distribution. It can be used for the analysis of data that has outliers. The Cauchy distribution can be derived as the ratio of two independent normal variates. It is a well-known distribution that can be applied in many fields such as biological sciences, applied mathematics, medicine, neural network, engineering, physics, econometrics, clinical trials, queuing theory, stochastic or time series modeling of descending failure rate life event or component and reliability. The cumulative distribution function (CDF) and probability density function (PDF) of Cauchy distribution having location parameter α and non-negative scale parameter $\beta > 0$ are expressed as,

$$F(x;\alpha,\beta)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\alpha}{\beta}\right)\ ;\ x\in\Re,\ \alpha,\beta>0 \quad (1)$$

and

$$f(x;\alpha,\beta)=\left\{\beta\pi\left[1+\left(\frac{x-\alpha}{\beta}\right)^2\right]\right\}^{-1}\ ;\ x\in\Re,\ \alpha,\beta>0 \quad (2)$$

The finite moment generating function (mgf) of the Cauchy distribution does not exist, because of this CLT does not hold. Further, the MLEs of its parameters are not the best because of the lack of a closed-form solution. Because of these reasons, the applicability of this distribution in modeling real-life data is doubtful/ unrealistic. Hence there is a need for modification of the Cauchy distribution to overcome the above-mentioned deficiencies, Rider (1957) has introduced the generalized Cauchy distribution, whose PDF is

$$f(x)=\frac{\Gamma(k)}{\lambda\,\Gamma(1/2)\Gamma(k-1/2)}\left[1+\left(\frac{x-\alpha}{\beta}\right)^2\right]^{-k}; \quad (3)$$
$$k\ge 1,\ \alpha,\beta>0,-\infty<x<\infty$$

To solve the problem of the non-existence of the MLEs and moments of Cauchy distribution a truncated Cauchy distribution was introduced by (Nadarajah & Kotz, 2006; Dahiya *et al.*, 2015) whose PDF is

$$f(x)=\frac{1}{\beta}\left[1+\left(\frac{x-\alpha}{\beta}\right)^2\right]^{-1}\left[\tan^{-1}\left(\frac{B-\alpha}{\beta}\right)-\tan^{-1}\left(\frac{A-\alpha}{\beta}\right)\right]^{-1};$$
$$-\infty<A\le x\le B,\alpha\in\Re,\beta>0$$

(4)

Also, Manoukian and Nadeau (1988) and Kravchuk (2005) have introduced a relation between Cauchy and Hyperbolic secant distribution. Ohakwe and Osu (2011) obtained a modified version of the Cauchy distribution. Another modification of the Cauchy distribution was suggested by (Alshawarbeh *et al.,* 2012; 2013) and (Eugene *et al.* 2002). Further, some generalizations of the Half- Cauchy distribution are introduced by using Kumasaswamy-G (Cordeiro & de Castro, 2011). Similarly, there are some half- Cauchy families which were namely Marshall- Olkin- half Cauchy, beta- half Cauchy, Kumaraswamy-half Cauchy put forwarded by Cordeiro and Lemonte (2011), Jacob and Jayakumar (2012), and Ghosh (2014), respectively.

In recent days, extensive study has been done to obtain models that fit survival data, which can be positively skewed, negatively skewed, and can have the unimodal hazard function. Rooks *et al.* (2010) has presented a two-parameter model that performs well with the survival data called Power Cauchy (PC) distribution. The PDF of the Power Cauchy distribution has a slightly thicker right tail than the other well-known two-parameter humped-shaped sub-model of the transformed beta family and it can be used for positively skewed data (Rooks *et al.*, 2010). The CDF and PDF of Power Cauchy distribution are,

$$F(x) = 2\pi^{-1}\tan^{-1}(\lambda x)^{\alpha};\; x>0,\;\alpha,\lambda>0 \text{ and} \quad (5)$$

$$f(x) = 2\pi^{-1}(\alpha\lambda)(\lambda x)^{\alpha-1}\left[1+(\lambda x)^{2\alpha}\right]^{-1};\; x>0. \quad (6)$$

respectively. The Hazard function of PC distribution is

$$h(x) = \frac{2\pi^{-1}(\alpha\lambda)(\lambda x)^{\alpha-1}\left[1+(\lambda x)^{2\alpha}\right]^{-1}}{1-2\pi^{-1}\tan^{-1}(\lambda x)^{\alpha}};\; x>0. \quad (7)$$

Chaudhary *et al.* (2020) have introduced truncated Cauchy power–inverse exponential distribution using truncated Cauchy power-G family of distribution. Further, Chaudhary *et al.* (2020a) have also defined a more flexible model called truncated Cauchy power–exponential model with decreasing, increasing, and up-side-down bathtub-shaped hazard function.

The major objective of this research is to introduce a wide applicable model to enhance the goodness-of-fit to the real-life data by inserting just one extra shape parameter. In this paper, we have used a power transformation approach to generate a new model called Exponentiated inverse Power Cauchy (EIPC) distribution. Further, researchers have illustrated some mathematical as well as statistical characteristics and properties of the new model. The contents of this article are systematized as follows. We have introduced the new distribution EIPC along with some distributional properties like the shape of the density, quantile function, survival and hazard rate function, random number generation, order statistics, and kurtosis and skewness in section 2. In section 3 we discuss the estimation method of the model parameters. The MLE method is used to estimate the parameters of the purposed distribution. For the MLE, we have constructed the asymptotic 95% confidence intervals for the parameters of EIPC distribution using the observed information matrix. In section 4 we illustrate a goodness-of-fit test and model adequacy test by considering a real data set. In section 5 a brief conclusion about the findings is presented.

NEW DISTRIBUTION DEVELOPMENT

In this section, we have introduced the new distribution named exponentiated inverse power Cauchy (EIPC) distribution and displayed some plots of its PDF and HRF. To define EIPC distribution we have first obtained the CDF of the inverse of PC distribution with shape parameter 'α' and scale parameter 'λ' using equation (5) is

$$F(x) = 1-2\pi^{-1}\tan^{-1}\left[\left(\frac{\lambda}{x}\right)^{\alpha}\right];\; x>0,\;\alpha,\lambda>0 \quad (8)$$

And its corresponding PDF is obtained as,

$$f(x) = 2\pi^{-1}\alpha\lambda^{\alpha}x^{-(\alpha+1)}\left[1+\left(\frac{\lambda}{x}\right)^{2\alpha}\right]^{-1};\; x>0,\alpha,\lambda>0 \quad (9)$$

This Inverse PC distribution is mainly related to the distribution introduced by Rooks et al. (2010) taking the random variable $Y = 1/X$ as the inverse random variable. Consider X to be a continuous random variable that follows EIPC distribution if it's CDF (using equation 8) with three parameters α, β and λ is

$$F(x) = \left[1-\frac{2}{\pi}tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta};(\alpha,\beta,\lambda)>0,x>0 \quad (10)$$

And its corresponding PDF can be defined as

$$f(x;\alpha,\beta,\lambda) = \frac{2}{\pi}\frac{\alpha\beta}{x}\left(\frac{\lambda}{x}\right)^{\alpha}\left\{1+\left(\frac{\lambda}{x}\right)^{2\alpha}\right\}^{-1}\times \left[1-\frac{2}{\pi}tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta-1};\alpha,\beta,\lambda>0,x>0 \quad (11)$$

Survival function of EIPC distribution

The survival function of the EIPC model is

$$S(x) = 1-\left[1-\frac{2}{\pi}tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta};\alpha,\beta,\lambda>0,x>0 \quad (12)$$

Hazard rate function (HRF) of EIPC distribution

The HRF of X can be obtained as

$$h(x) = \frac{\frac{2\alpha\beta}{\pi\, x}\left(\frac{\lambda}{x}\right)^{\alpha}\left\{1+\left(\frac{\lambda}{x}\right)^{2\alpha}\right\}^{-1}\left[1-\frac{2}{\pi}tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta-1}}{1-\left[1-\frac{2}{\pi}tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta}} \quad (13)$$

The particular case of the EIPC distribution:

If $\alpha = 1,\ \beta = 1, and\ \lambda = 1$ in equation (11) the EIPC model reduces to two times the standard Cauchy distribution.

The cumulative hazard function (CHF)

The CHF of the EIPC model is

$$H(x) = \int_{-\infty}^{x} h(x)dx$$

$$= -log[1 - F(x)]$$

$$= -log\left[1 - \left[1 - \frac{2}{\pi} tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta}\right] \quad (14)$$

Failure rate average (FRA):

$$FRA(x) = \frac{H(x)}{x}$$

$$= -\frac{1}{x} log\left[1 - \left[1 - \frac{2}{\pi} tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta}\right]; x > 0 \quad (15)$$

Where the term H(x) is the CHF of the EIPC model. The graph of PDF and HRF of the EIPC model for various values of α, β, and λ are displayed in Fig. 1.

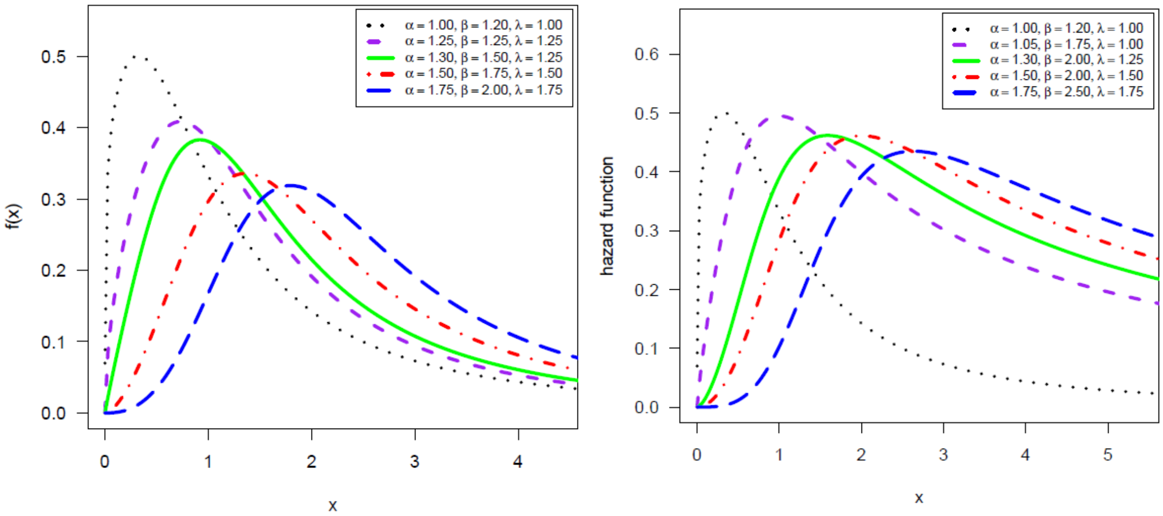

Figure 1. Graph of PDF (left panel) and HRF (right panel) for various values of α, β, and λ of EIPC distribution

STATISTICAL PROPERTIES OF EIPC DISTRIBUTION

The Quantile Function of EIPC distribution:

The continuous random variable X follows $EIPC(\alpha, \beta, \lambda)$ with CDF *F(x)* then the quantile function is the inverse of the CDF and it is calculated as

$$Q(p) = F^{-1}(p) = \min\{x \in \Re : F(x) \geq p\} \quad ; p \in (0,1)$$

$$Q(p) = \lambda\left[\tan\left\{\frac{\left(1 - p^{1/\beta}\right)\pi}{2}\right\}\right]^{-1/\alpha} \quad ; 0 < p < 1 \quad (16)$$

The random number generating function

The pseudo-random number can be produced from $EIPC\ (\alpha, \beta, \lambda)$ by

$$x = \lambda\left[\tan\left\{\frac{\left(1 - b^{1/\beta}\right)\pi}{2}\right\}\right]^{-1/\alpha} \quad ; 0 < b < 1 \quad (17)$$

Where b follows the uniform distribution $U(0,1)$ distribution.

Mode of EIPC distribution

A mode is the most repetitive value of the probability distribution of the given PDF. The necessary and sufficient conditions for calculating the mode are,

$\frac{df(x)}{dx} = 0$ and $\frac{d^2 f(x)}{dx^2} < 0$ respectively. Since $f(x) > 0$, the model value of the proposed distribution is calculated by solving the equation

$$\frac{2\alpha(\beta - 1)\lambda^{\alpha} x^{-2\alpha}}{\pi - 2\tan^{-1}(\lambda / x)^{\alpha}} + \frac{(\lambda / x)^{2\alpha}(2\alpha - 1) - 1}{1 + (\lambda / x)^{2\alpha}} = 0 \quad (18)$$

Manually it is difficult to solve the equation (18) because of nonlinear, so via the Newton-Raphson technique, we can solve it numerically which gives the model value of the proposed distribution.

Expression for Skewness and Kurtosis:

The relation that measures the Skewness which is based on quantile is,

$$Skewness(B) = \frac{Q(3/4) + Q(1/4) - 2Q(0.5)}{Q(3/4) - Q(1/4)} \quad (19)$$

Expression for Kurtosis:

The relation that measures the kurtosis based on octiles (Moors, 1988) can be computed as

$$Kurtosis(M) = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(3/4) - Q(1/4)} \quad (20)$$

For the study of the nature and behavior of the proposed distribution, we have generated the random samples of size 100 each using equation (17) for EIPC distribution for ten different combinations for triplet $\Psi = (\alpha, \beta, \lambda)$.

In Table 1 we have presented different measures of central tendencies and dispersion such as mean, median, mode, skewness (using equation 19), and kurtosis (using equation 20) of the EIPC distribution. Case-I shows that $\alpha \geq 0.5$ mean, median, mode, skewness, and kurtosis are decreased. Since skewness is positive in cases I, II, and III, so the proposed distribution is positively skewed. The measure of kurtosis reveals that the proposed distribution is leptokurtic at the beginning and gradually it changes to platykurtic as increased the values α. Case-II shows that as $\beta \geq 0.5$ mean, median, modes, skewness, and kurtosis increase. For the case-III, the mean, median, and modes increase as the increase in value of λ but skewness and kurtosis do not alter. Also in all three cases, we noticed that $mean > median > \mathrm{mod}e$, so we conclude that the EIPC distribution is positively skewed.

Table 1. Mean, median, mode, skewness, and kurtosis for various values of parameters

α	β	λ	mean	median	Mode	skewness	kurtosis
				Case-I			
0.5	**5**	**10**	38207.309	195.818	76.68382	9.7056	95.7023
1	**5**	**10**	174.6615	44.2513	27.69184	7.9037	68.7568
1.5	**5**	**10**	49.7329	26.9536	19.71974	5.8956	42.3526
2	**5**	**10**	30.2088	21.036	16.64087	4.5127	26.9539
2.5	**5**	**10**	23.221	18.1289	15.02937	3.6536	18.6984
5	**5**	**10**	14.6869	13.4643	12.25943	2.12	7.0134
10	**5**	**10**	12.0213	11.6036	11.07223	1.5047	3.7048
20	**5**	**10**	10.9434	10.772	10.52247	1.2347	2.5661
30	**5**	**10**	10.6151	10.5083	10.34535	1.1496	2.2515
40	**5**	**10**	10.4562	10.3788	10.25791	1.1079	2.1054
				Case-II			
5	**0.5**	**10**	8.5363	8.1319	6.656049	0.8625	2.3877
5	**1**	**10**	10.4409	9.7905	8.653295	1.4418	4.7041
5	**1.5**	**10**	11.4848	10.6696	9.601269	1.7565	5.7961
5	**2**	**10**	12.2196	11.2956	10.23097	1.9156	6.3388
5	**2.5**	**10**	12.7964	11.7938	10.71249	2.0012	6.6299
5	**5**	**10**	14.6869	13.4643	12.25943	2.1200	7.0134
5	**10**	**10**	16.8180	15.3887	13.99657	2.1351	7.0230
5	**20**	**10**	19.2694	17.6219	16.0063	2.1255	6.9548
5	**30**	**10**	20.8753	19.0888	17.32838	2.1190	6.9181
5	**40**	**10**	22.0993	20.2074	18.33785	2.1149	6.8966
				Case-III			
10	**5**	**0.5**	0.6011	0.5802	0.55361	1.5047	3.7048
10	**5**	**1**	1.2021	1.1604	1.10722	1.5047	3.7048
10	**5**	**1.5**	1.8032	1.7405	1.660835	1.5047	3.7048
10	**5**	**2**	2.4043	2.3207	2.214446	1.5047	3.7048
10	**5**	**2.5**	3.0053	2.9009	2.768058	1.5047	3.7048
10	**5**	**5**	6.0107	5.8018	5.536116	1.5047	3.7048
10	**5**	**10**	12.0213	11.6036	11.07223	1.5047	3.7048
10	**5**	**20**	24.0427	23.2072	22.14446	1.5047	3.7048
10	**5**	**30**	36.0640	34.8108	33.2167	1.5047	3.7048
10	**5**	**40**	48.0854	46.4144	44.28893	1.5047	3.7048

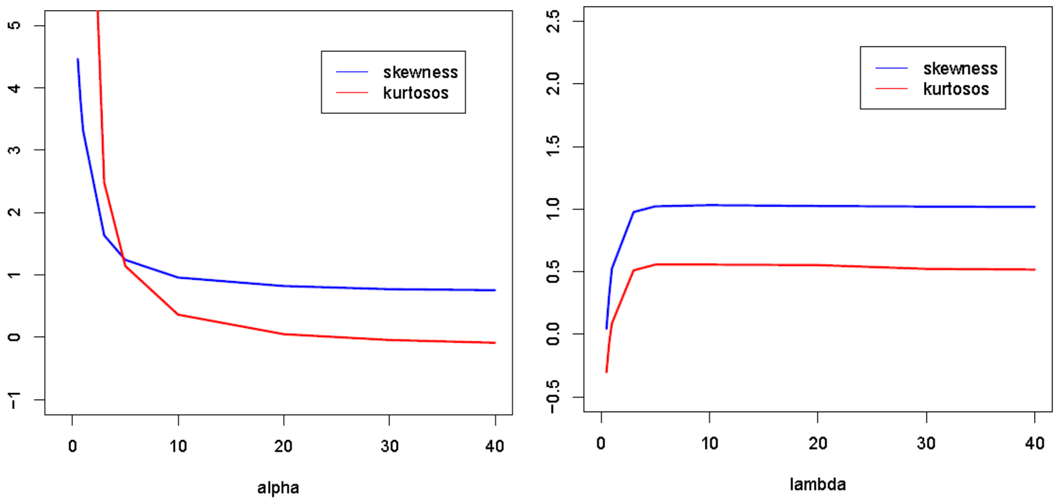

Figure 2. Skewness and Kurtosis corresponding to the various values of the parameters α and λ

Linear representation of EIPC distribution

The PDF and CDF of EIPC distribution are extended by using the generalized binomial theorem as

$$(1-a)^n = \sum_{i=0}^{\infty}(-1)^i \binom{n}{i} a^i \quad for, n>0 \qquad (21)$$

$$(1+a)^{-n} = \sum_{k=0}^{\infty}(-1)^k \binom{n+k-1}{k} a^k, \text{ for } |a|<1, n>0 \qquad (22)$$

Applying equation (21) in equation (11) the PDF of EIPC distribution is expressed as

$$f(x) = C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta_{ij} x^{-(2i\alpha+\alpha+1)}\left[\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^j \qquad (23)$$

where $C = \frac{2}{\pi}\alpha\beta\lambda^{\alpha}$ and

$$\theta_{ij} = (-1)^{i-j}\binom{\beta-1}{j}\lambda^{2i\alpha}(2\pi^{-1})^j$$

also the CDF is

$$[F(x)]^h = \left[1-2\pi^{-1}\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{\beta h}$$

$$= \sum_{j=0}^{\infty}(-1)^j\binom{\beta h}{j}\left[2\pi^{-1}\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^j$$

$$= \sum_{j=0}^{\infty}(-2\pi^{-1})^j\binom{\beta h}{j}\left[\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^j$$

Order Statistics

Order statistics can be used in many fields of probability theory and applied statistics. Hence we explore some properties and characteristics of order statistics for the proposed distribution. Consider $X_1,\dots,X_n$ be n independently and identically distributed random variates, each with CDF F(x). If these variables are sorted in increasing order of magnitude and they are written as $X_{(1)} \le \dots \le X_{(n)}$. The term $X_{(r)}$ is called the r^th^ order statistic, where $r = 1,2,\dots,n$. Suppose $X_{r:n}$ represents the r^th^ order statistic and $f_{r:n}$ denote the PDF of r^th^ order statistic for $X_1,\dots,X_n$ be n identically and independently (IID) distributed random variables from CDF F(x) and can be defined as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\left[1-F(x)\right]^{n-r}\left[F(x)\right]^{r-1}$$

$$= \frac{n!}{(r-1)!(n-r)!} f(x)\sum_{j=1}^{n-r}\binom{n-r}{j}\left[F(x)\right]^{j+r-1}$$

$$= \frac{n!}{(r-1)!(n-r)!}\frac{2}{\pi}\frac{\alpha\beta}{x}\left(\frac{\lambda}{x}\right)^{\alpha}\left\{1+\left(\frac{\lambda}{x}\right)^{2\alpha}\right\}^{-1}\sum_{j=1}^{n-r}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}C_{jkl}\left[2\pi^{-1}\tan^{-1}\left\{\left(\frac{\lambda}{x}\right)^{\alpha}\right\}\right]^{k+l} \qquad (24)$$

Where $C_{jkl} = (-1)^{k+l}\binom{n-r}{j}\binom{\beta j+\beta r-\beta}{k}\binom{\beta-1}{l}$.

ESTIMATION METHOD OF THE PARAMETERS OF EIPC DISTRIBUTION

In this segment, we discuss the maximum likelihood method for estimating the constants of the EIPC distribution and we use them to obtain the confidence intervals. Let $\underline{x} = \left(x_1,\ldots,x_n\right)$ be a non-negative random sample of size 'n' taken from $EIPC(\alpha,\ \beta,\ \lambda)$ then the log-likelihood function is

$$l = n\ln(2/\pi) + n\ln(\alpha) + n\ln(\beta) - \sum_{i=1}^{n}\ln(x_i) + n\alpha\ln(\lambda) - \alpha\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\ln\left\{1+u_i^{2\alpha}\right\} + (\beta-1)\sum_{i=1}^{n}\ln\left[1-\frac{2}{\pi}\tan^{-1}\left\{u_i^{\alpha}\right\}\right] \tag{25}$$

taking $u_i = \lambda / x_i$ and after differentiation (25) w.r.t. α, β *and* λ, gives

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + n\ln(\lambda) + \sum_{i=1}^{n}\ln(x_i) - 2\sum_{i=1}^{n}\frac{u_i^{2\alpha}\ln u_i}{1+u_i^{2\alpha}} - 2(\beta-1)\sum_{i=1}^{n}\frac{u_i^{\alpha}\ln u_i}{\left\{1+u_i^{2\alpha}\right\}\left\{\pi - 2\tan^{-1}\left(u_i^{\alpha}\right)\right\}}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left[1-\frac{2}{\pi}\tan^{-1}\left\{u_i^{\alpha}\right\}\right]$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n\alpha}{\lambda} - \frac{2\alpha}{\lambda}\sum_{i=1}^{n}\frac{u_i^{2\alpha}}{\left[1+u_i^{2\alpha}\right]} + \frac{2\alpha(\beta-1)}{\lambda}\sum_{i=1}^{n}\frac{u_i^{\alpha}}{\left\{1+u_i^{2\alpha}\right\}\left\{2\tan^{-1}\left(u_i^{\alpha}\right) - \pi\right\}}$$

By solving these non-linear equations for the parameters of the EIPC distribution (α, β, λ) by setting them to zero we will find the ML estimators of the EIPC distribution. Computer software like R, Mathematica, Matlab, etc can be used to solve them manually. Let the vector of parameter $\underline{\iota} = (\alpha, \beta, \lambda)$ and the corresponding MLE of $\underline{\iota}$ as $\hat{\underline{\iota}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ then $(\hat{\underline{\iota}} - \underline{\iota}) \to N_3\left[0, \left(K(\underline{\iota})\right)^{-1}\right]$ follows the normal distribution, where $K\left(\underline{\iota}\right)$ is the Fisher's information matrix obtained by,

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}$$

where $G_{11} = \frac{\partial^2 l}{\partial \alpha^2},\ G_{12} = \frac{\partial^2 l}{\partial \alpha \partial \beta},\ G_{13} = \frac{\partial^2 l}{\partial \alpha \partial \lambda}$

$$G_{21} = \frac{\partial^2 l}{\partial \beta \partial \alpha},\ G_{22} = \frac{\partial^2 l}{\partial \beta^2},\ G_{23} = \frac{\partial^2 l}{\partial \beta \partial \lambda}$$

$$G_{31} = \frac{\partial^2 l}{\partial \lambda \partial \alpha},\ G_{32} = \frac{\partial^2 l}{\partial \beta \partial \lambda},\ G_{33} = \frac{\partial^2 l}{\partial \lambda^2}$$

Taking second-order differentiation of (25) we get

$$G_{11} = -\frac{n}{\alpha^2} - 4\sum_{i=1}^{n}\frac{u_i^{2\alpha}\left[\ln u_i\right]^2}{\left[1+u_i^{2\alpha}\right]^2} + 2\pi\left(\beta-1\right)\sum_{i=1}^{n}\frac{u_i^{\alpha}\left[\ln u_i\right]^2\left[\left\{u_i^{2\alpha}-1\right\} - 2u_i^{\alpha}\right]}{\left\{1+u_i^{2\alpha}\right\}\left\{\pi - 2\tan^{-1}\left\{u_i^{\alpha}\right\}\right\}^2} - 4\left(\beta-1\right)\sum_{i=1}^{n}\frac{u_i^{\alpha}\left[\ln u_i\right]^2(u_i^{2\alpha}-1)\tan^{-1}u_i^{\alpha}}{\left\{1+u_i^{2\alpha}\right\}\left\{\pi - 2\tan^{-1}\left\{u_i^{\alpha}\right\}\right\}^2}$$

$$G_{22} = -\frac{n}{\beta^2}$$

$$G_{33} = -\frac{n\alpha}{\lambda^2} - \frac{2\alpha}{\lambda^2}\sum_{i=1}^{n}\frac{u_i^{2\alpha}\left[1-2\alpha+u_i^{2\alpha}\right]}{\left[1+u_i^{2\alpha}\right]} - \frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{u_i^{\alpha}\left\{(2\alpha+2)u_i^{2\alpha}-2\alpha+2\right\}\tan^{-1}u_i^{\alpha}}{\left\{1+u_i^{2\alpha}\right\}^2\left\{2\tan^{-1}u_i^{\alpha}-\pi\right\}^2}$$
$$-\frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{u_i^{\alpha}\left\{\pi(-\alpha-1)u_i^{2\alpha}+2\alpha u_i^{\alpha}+\pi(\alpha-1)\right\}}{\left\{1+u_i^{2\alpha}\right\}^2\left\{2\tan^{-1}u_i^{\alpha}-\pi\right\}^2}$$

$$G_{12} = G_{21} = -2\sum_{i=1}^{n}\frac{u_i^{\alpha}\ln u_i}{\left(1+u_i^{2\alpha}\right)\left(\pi-2\tan^{-1}u_i^{\alpha}\right)}$$

$$G_{23} = G_{32} = -\frac{2\alpha}{\lambda}\sum_{i=1}^{n}\frac{u_i^{\alpha}}{\left(1+u_i^{2\alpha}\right)\left(2\tan^{-1}u_i^{\alpha}-\pi\right)}$$

$$G_{13} = G_{31} = \frac{n}{\lambda} + \frac{2\alpha}{\lambda}\sum_{i=1}^{n}\frac{u_i^{2\alpha}\left[1+2\alpha\ln u_i+u_i^{2\alpha}\right]}{\left[1+u_i^{2\alpha}\right]} + 2\pi\alpha(\beta-1)\sum_{i=1}^{n}\frac{u_i^{\alpha}\left(u_i^{2\alpha}\ln u_i-\ln u_i\right)}{\left\{1+u_i^{2\alpha}\right\}^2\left\{\pi-2\tan^{-1}u_i^{\alpha}\right\}^2} +$$
$$2(\beta-1)\sum_{i=1}^{n}\frac{u_i^{\alpha}\tan^{-1}u_i^{\alpha}\left(\ln u_i-\ln u_i^{2\alpha}\right)}{\left\{1+u_i^{2\alpha}\right\}^2\left\{\pi-2\tan^{-1}u_i^{\alpha}\right\}^2} + 2(\beta-1)\sum_{i=1}^{n}\frac{u_i^{\alpha}\left[\pi\left\{-u_i^{2\alpha}-1\right\}-2u_i^{\alpha}\ln u_i\right]}{\left\{1+u_i^{2\alpha}\right\}^2\left\{\pi-2\tan^{-1}u_i^{\alpha}\right\}^2}$$
$$+4(\beta-1)\sum_{i=1}^{n}\frac{u_i^{\alpha}\left(u_i^{2\alpha}\tan^{-1}u_i^{\alpha}+\tan^{-1}u_i^{2\alpha}\right)}{\left\{1+u_i^{2\alpha}\right\}^2\left\{\pi-2\tan^{-1}u_i^{\alpha}\right\}^2}$$

Practically $\underline{\iota}$ is unknown hence it is inadequate that the MLE has a variance $\left(G\left(\underline{\iota}\right)\right)^{-1}$ so we approximate this variance by putting the estimated value of the model parameters. Here $G\left(\underline{\iota}\right)$ represents the Fisher's information matrix. The observed information matrix is calculated by maximizing the likelihood function through the Newton-Raphson algorithm and the var-cov matrix can be obtained as,

$$\left(G(\underline{\iota})\right)^{-1} = \begin{pmatrix} V(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\alpha},\hat{\beta}) & V(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\alpha}) & cov(\hat{\lambda},\hat{\beta}) & V(\hat{\lambda}) \end{pmatrix}. \quad (26)$$

Hence using the matrix (26) we can construct the 100(1-ϑ) % asymptotic CI for α, β, and λ as,

$$\hat{\alpha} \pm Z_{\vartheta/2}SE(\hat{\alpha}),\ \hat{\beta} \pm Z_{\vartheta/2}SE(\hat{\beta}) \text{ and } \hat{\lambda} \pm Z_{\vartheta/2}SE(\hat{\lambda}) \quad (27)$$

APPLICATIONS TO REAL DATASET

In this section, we illustrate the capability and applicability of the proposed model using a real data set used by former researchers. The dataset contains the failure times (in hours) of 59 conductors originally used by (Nelson & Doganaksoy, 1995). The data set doesn't contain any censored observations.

6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

Estimation of the model parameters

In this subsection, the parameters of the proposed distribution are obtained using the MLE method for the above real data set. Maximizing the log-likelihood function (25) we have computed the MLEs employing the AdequacyModel package in R software (R Core Team, 2021; Dalgaard, 2008). We have presented the MLEs along with their standard errors (SE) and 95% asymptotic confidence interval (ACI).

Table 2. MLE with SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	7.2367	1.4256	(4.4425, 10.0309)
beta	0.7421	0.2592	(0.2341, 1.2501)
lambda	7.2478	0.4694	(6.3278, 8.1678)

The Hessian variance-covariance matrix is,

$$\text{var}-\text{cov } matrix = \begin{pmatrix} V(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\lambda}) \\ \text{cov}(\hat{\alpha},\hat{\beta}) & V(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\lambda}) \\ \text{cov}(\hat{\lambda},\hat{\alpha}) & \text{cov}(\hat{\lambda},\hat{\beta}) & V(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} 2.0324256 & -0.29502216 & 0.4654014 \\ -0.2950222 & 0.06717508 & -0.1088797 \\ 0.4654014 & -0.10887975 & 0.2202969 \end{pmatrix}$$

Model validation and goodness-of-fit

The Kolmogorov-Smirnov (KS) test is employed to substantiate the validity of the proposed model. This test is used to compare the distance between the empirical distribution function (EDF) and the fitted distribution function. In Fig. 3 (left panel) the graph of the KS plot exhibits that our model EIPC fits the dataset very nicely. Also, we have obtained the $p-value = 0.9852$ for the KS test to our EIPC distribution, which indicates a better fit for the real data set taken under study. To dig up the behavior and shape of the hazard function, we have plotted the total-time-on test (TTT) plot of the empirical version of the scaled TTT transform of the data set (Fig. 3, right panel) (Aarset, 1987). It is seen that the graph of the TTT plot is concave; which indicates that the hazard function of the failure times of conductors is increasing. Further, in Fig. 4 the probability-probability (P-P) and the quantile-quantile (Q-Q) plots are displayed for the verification of empirical distribution versus theoretical distribution fitting. The graphs of Q-Q and P-P have revealed that the empirical distribution versus theoretical distribution is strong for EIPC distribution. Hence, we hope that EIPC distribution will be an alternative model for real data analysis in the future.

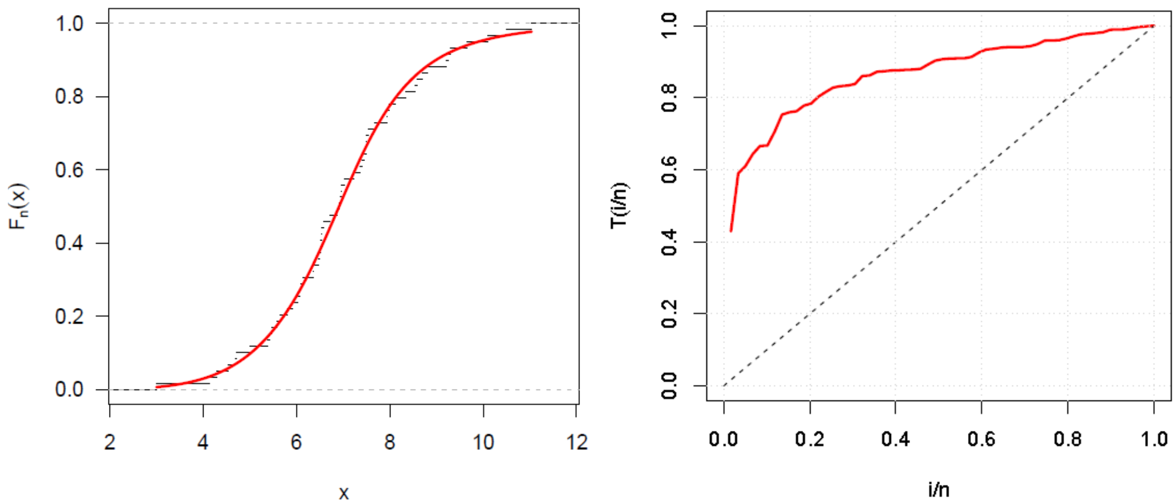

Figure 3. KS plot for Empirical vs. Fitted CDF (left panel) of EIPC distribution and TTT-plot (right panel) of the data under study

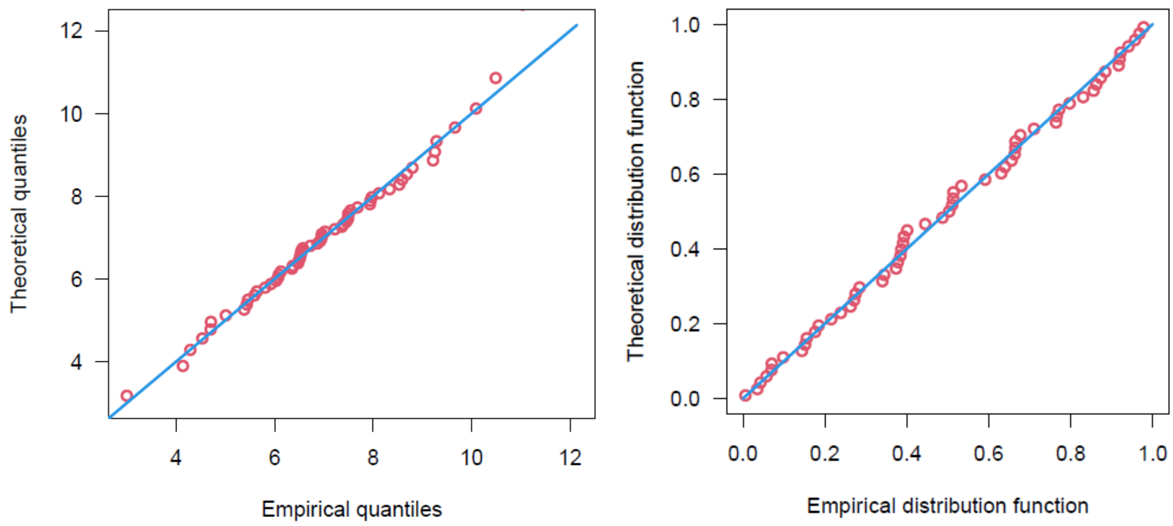

Figure 4. The graph of the Q-Q plot (left panel) and P-P plot (right panel)

Model Comparison

In this sub-section, the proposed distribution EIPC is mainly based on the Cauchy distribution so we have considered four well-known distributions for comparison purposes where three of them are related to Cauchy based distributions, which are as follows

a) Power Cauchy distribution

The PDF of Power Cauchy (PC) distribution (Rooks *et al.,* 2010) with two parameters (α, λ) is

$$f_{PC}(x)=2\pi^{-1}(\alpha\lambda)(\lambda x)^{\alpha-1}\left[1+(\lambda x)^{2\alpha}\right]^{-1};\ x>0,\ \alpha,\lambda>0$$

Lindley half-Cauchy distribution

The PDF of Lindley half-Cauchy (LHC) distribution (Chaudhary & Kumar, 2020) with two parameters is

$$f_{LHC}(x)=\frac{2}{\pi}\left(\frac{\theta^2}{1+\theta}\right)\left(\frac{\lambda}{\lambda^2+x^2}\right)\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right\}^{\theta-1}\times$$
$$\left\{1-\ln\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\};\ \theta,\lambda>0,\ x>0$$

Half-Cauchy distribution

Paradis *et al.* (2002) used the half-Cauchy distribution (HC) with PDF

$$f_{HC}(x)=\frac{2}{\pi}\left(\frac{\lambda}{\lambda^2+x^2}\right),\ \lambda>0,\ x>0$$

b) Weibull Extension distribution

The PDF of the Weibull extension (WE) distribution (Tang *et al.*, 2003) with three parameters (α,β,λ) is

$$f_{WE}(x;\alpha,\beta,\lambda)=\lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(\frac{x}{\alpha}\right)^{\beta}\times$$
$$\exp\left\{-\lambda\alpha\left(\exp\left(\frac{x}{\alpha}\right)^{\beta}-1\right)\right\}\ ;\ x>0,(\alpha,\beta,\lambda)>0$$

For the comparison we have fitted the above mentioned four distributions and also calculated log-likelihood (-LL), Akaike information criterion (AIC), Corrected Akaike Information Criterion (AICC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) statistic to verify the potentiality of the EIPC model. To compute the above criteria we have used the following expressions

$$AIC=-2l(\hat{\theta})+2p$$
$$BIC=-2l(\hat{\theta})+p\log(n)$$
$$AICC=\frac{2p(p+1)}{n-p-1}+AIC$$
$$HQIC=-2l(\hat{\theta})+2p\log\left[\log(n)\right]$$

where p is the parameters contained in a model and n is the sample size under consideration. From table 3 we found that the proposed model EIPC is good as compared to WE, LHC, and HC models and nearly similar compared to the PC distribution taken under study.

Table 3. Goodness of fit statistics

Model	-LL	AIC	BIC	AICC	HQIC
EIPC	111.7294	229.4588	235.6914	229.8799	231.8918
PC	112.0913	228.1826	232.3376	228.3968	229.8045
WE	113.5215	233.0430	239.2756	233.4793	235.4795
LHC	170.3326	344.6653	348.8204	344.8796	346.2873
HC	182.1991	366.3982	368.4757	366.4684	367.2092

By using the MLE method we have calculated the parameter of all of the above models taken for comparison. Also, to assess the potentiality of the EIPC distribution we have calculated the Anderson-Darling (W), Kolmogorov-Simnorov (KS), and the Cramer-Von Mises (A^2) statistics presented in Table 4. These statistics are extensively applied to evaluate the non-nested models and also used to show how closely a specific CDF fits the EDF of a given data set. The mathematical expression to obtain these statistics are

$$KS=\max_{1\le i\le n}\left(d_i-\frac{i-1}{n},\frac{i}{n}-d_i\right)$$
$$W=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i+\ln(1-d_{n+1-i})\right]$$
$$A^2=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{(2i-1)}{2n}-d_i\right]^2$$

where x_i's are the ordered samples and $d_i = CDF(x_i)$.

From the Table 4, we have noticed that the EIPC model attains the lowest value of the test statistic and the highest *p*-value so we confirm that the proposed model acquires a superior fit as compared to WE, LHC, and HC models and is nearly similar as compared to PC distribution and can produce more reliable and consistent results. In Fig. 5, the histogram with the fitted PDF (left panel) and the empirical CDF with theoretical CDF (right panel) for the dataset under study are presented. Therefore, for the given data set it is found that the proposed distribution gets a better fit than WE, LHC, and HC models and is nearly similar as compared to the PC model selected for comparison.

Table 4. The goodness-of-fit statistic with p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
EIPC	0.0569(0.9852)	0.1437(0.999)	0.0212(0.9961)
PC	0.0480(0.9982)	0.1780(0.9953)	0.0199(0.9973)
WE	0.1063(0.4852)	0.6648(0.5886)	0.1139(0.5222)
LHC	0.4151(1.01e-09)	15.555(1.01e-05)	3.2399(7.18e-09)
HC	0.3517(4.95e-07)	14.226(1.01e-05)	2.8148(9.60e-08)

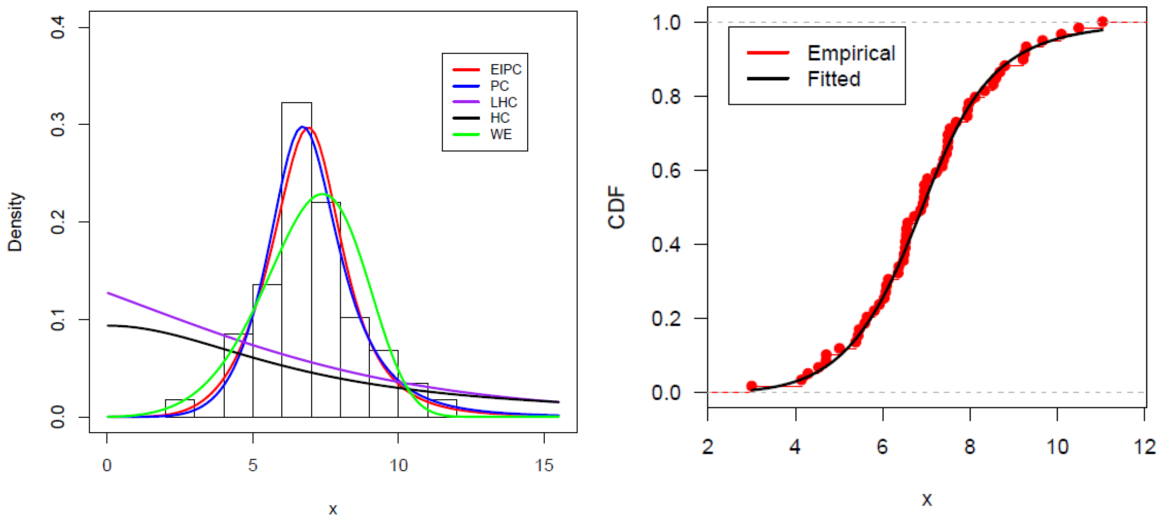

Figure 5. The PDF of fitted density with histogram (left panel) and estimated CDF with empirical CDF (right panel)

CONCLUSIONS

In this study, we have suggested a new continuous distribution named exponentiated inverse power Cauchy (EIPC) distribution. We have discussed some chief characteristics and properties of the new model like the shapes of the PDF, CDF, and hazard rate functions; also we derive the expressions for survival function, quantile function, reverse hazard rate function, skewness, kurtosis measures, and order statistics. The MLE method is applied to estimate the parameters of the proposed model. A real lifetime data set is taken to investigate the suitability and applicability of the EIPC distribution. By comparing exponentiated inverse power Cauchy distribution with some other lifetime models taken into consideration, it is concluded that the proposed distribution performs well and provides a better fit as compared to WE, LHC, and HC models and is nearly similar as compared to the PC model. We expect that this new model may be a choice in the areas of probability theory, survival analysis, and applied statistics.

CONFLICT OF INTERESTS

The author declares no conflict of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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