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Research Article

AN FPTAS FOR QUICKEST MULTI-COMMODITY CONTRAFLOW PROBLEM WITH ASYMMETRIC TRANSIT TIMES

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ABSTRACT

One of the challenges in operations research is to route numerous commodities from particular supply locations to the corresponding demand points across the lanes of a network infrastructure while maintaining capacity restrictions. The quickest multi-commodity flow problem would be one of those that reduces the time it takes to complete the process. Reorienting lanes toward demand sites can increase outbound lane capacity. The quickest multi-commodity contraflow problem is NP-hard computationally. We use a Δ-condensed time-expanded graph to propose an FPTAS for this problem by including the lane reversal technique. We look into asymmetric transit times on anti-parallel arcs to address the unequal road conditions and flow dependency.

Keywords: Asymmetric transit times; contraflow; network flow; quickest multi-commodity; Δ-condensed

2020 Mathematics Subject Classification. Primary: 90B10, 90C27, 68Q25; Secondary: 90B06, 90B20.

INTRODUCTION

The multi-commodity network flow challenge involves delivering diverse commodities from specified source nodes to corresponding sink nodes while staying within arc capacity constraints. Railway networks, message routing in Telecommunications, industrial planning, logistics, supply chains for essential products and pharmaceuticals during catastrophes are only a few examples of network routing difficulties that may be considered as multi-commodity flow problems. The network topology is characterized as a road network that corresponds to the transshipment of various commodities, with nodes being distribution centers, demand sites, intersections of road segments, and arcs being links between nodes. The starting and ending destinations of commodities are called supply and demand nodes, respectively. The collection of items that have been carried via a network is referred to as flow. Capacity and trip times are assigned to the arcs in networks having temporal dimensions. For further information see (Assad, 1978; Kennington, 1978; Ahuja et al., 1993; Wang, 2018; Salimifard & Bigharaz, 2020).

Six decades ago, Ford and Fulkerson (1962) were credited with creating network flow over time. The opposite of this problem is the quickest flow problem, in which supplies and demands at the supply and demand points are known, and the goal is to discover the shortest time to meet demand. By performing a binary search on the maximum flow computation of Ford and Fulkerson (1962), authors in Burkard et al. (1993) discovered the first polynomial-time solution to the quickest flow problem.

By applying a parametric search strategy to the minimum cost flow problem, they reduce the complexity of the problem and established time bounds depending on the input size only. This problem was expanded to include multi-source and multi-sink scenarios. However, one extension is the quickest transshipment problem, wherein the supply and demand vectors at the endpoints are provided, and the aim is to create a dynamic flow that fulfills all supplies and demands in the minimum amount of time.

Multi-commodity flow problems are more difficult to solve than problems with the same type of flows. Even with series-parallel networks or flow with only two commodities, according to Hall et al. (2007). As a result, the quickest multi-commodity flow problem is NP-hard, whether with or without intermediate node storage and a simple flow path. Fleischer and Skutella (2002, 2007) offered two approximation approaches to solve this problem due to NP-hardness. The first is length-bounded flow, while the second involves discretizing a bigger time step rather than a single time step. Lozovanu and Fonoberova (2006) and Kappmeir (2015) employed a time-expanded network to address maximum multi-commodity flow over time, and Kappmeier (2015) extended their solution to multi-source single-sink multi-commodity earliest arrival transshipment difficulties with pseudo-polynomial time complexity.

The flipping of arc configurations to enhance capacity and improve traffic flow is known as lane reversal. For two-terminal maximum and quickest flow problems, Rebennack et al. (2010) provided models and highly polynomial-time algorithms. These lane reversals start at the beginning and are fixed as time goes on. The fundamental purpose of partial lane reversals is to utilize unused arc capacity on a network. Pyakurel et al. (2019) proposed a partial lane reversal technique that maximizes flow value by only flipping essential arc capacity. The capacity of unused arcs might be utilized for logistical help and facility placement in an emergency.

Nath et al. (2021) modified the algorithm of Rebennack et al. (2010) and solved the dynamic contraflow problems with non-symmetric transit times on anti-parallel arcs within the same time complexity. Gupta et al. (2021a, 2021b) extended the approach of Nath et al. (2021) to lexicographic flow, earliest arrival transshipment and generalized flow problems, and provided the solution for single-commodity. The approximate solution of the quickest multi-commodity flow problem was developed in Fleisher and Skutella (2002, 2007), using a T-length bounded function and Δ-condensed time-extended networks. Dhamala et al. (2020) and Gupta et al. (2020) introduced a partial lane reversal approach with symmetric transit times in this problem.

We present a fully polynomial-time approximation scheme (FPTAS) in this research that uses Δ-condensed time-expanded networks to solve the quickest multi-commodity contraflow (QMCCF) problem in the case of asymmetric transit times on anti-parallel arcs. The time span for transshipping commodities from supply nodes to demand nodes is reduced by using the lane reversal approach in a routing problem. The most important outcome of this research is the minimization of delivery time.

The following is a breakdown of how the paper is structured. The second part below contains the article's basic notations and models. In the third part, the QMCCF problem with asymmetric transit times on anti-parallel arcs is introduced. We provide an FPTAS in this part that provides a fully polynomial-time approximate solution to the problem. The final portion concludes the paper.

PRELIMINARIES AND MATHEMATICAL MODEL

To fulfill the complete demand for each commodity, the multi-commodity flow problem includes transferring numerous commodities from their respective supply points to their corresponding demand points throughout a given transportation network. We describe appropriate denotations and mathematical formulations for this problem, in which arc reversals can minimize the traversal time and increase flow value by switching their orientations as needed.

QMCCF MODEL

Let us consider a network architecture $G = (V, A, K, u, \tau, b_i, S_+, S_-, T)$, where V stands for the set of nodes, and A set of arcs, with set of commmodity K having k number of commodities. The number of nodes and arcs are denoted by n and m, respectively. Any object $i \in K$ with demand b_i is shipped via its origin-destination pair (s_i, t_i), with $s_i \in S_+ \subset V$ and $t_i \in S_- \subset V$. A travel time function $\tau : A \rightarrow R^+$ quantifies the time it takes to transship the flow from the beginning point x to the terminal point y of arc $e = (x, y)$, and the capacity function $u: A \rightarrow R^+$ controls the flow of commodities on each arc $e = (x, y)$. The time period T is specified in advance in both discrete and continuous-time settings, is represented by $\boldsymbol{T} = \{0, 1, \ldots, T\}$ and $\boldsymbol{T} = [0, T + 1)$, respectively.

The sets $\delta^+(x) = \{(x,.) \mid . \in V\}$ and $\delta^-(x) = \{(.,x) \mid . \in V\}$ represent arcs passing and joining node x, respectively, such that $\delta^+(S_-) = \delta^-(S_+) = \emptyset$, except in the contraflow network. The auxiliary network for a given network G is denoted by $G^a = (V, A^a, K, u^a, \tau^a, b_i, S_+, S_-, T)$, with arcs having no direction in $A^a = \{(x, y): (x, y) \text{ or } (y, x) \in A\}$, where $e^r = (y, x)$ is the backward arc of $e = (x, y)$. The sum of the capacities of forward and backward arcs is the capacity of arcs in the network G^a. The capacity of arc $u_e = 0$ if $e \notin A$. The transit time of auxiliary arc τ_a is the capacity of non-flipped arc. In the case of a single arc, we assume $\tau_a = \tau_e = \tau_{e^r}$.

$G = (V, A, K, u, b_i, S_+, S_-)$ represents the static network without the time dimension. The function $f : A \rightarrow R^+$ is the static multi-commodity flow. Many useful features derived from static network topology serve as foundational tools for the majority of real-world dynamic flow challenges.

A multi-commodity flow over times Φ for a given network G with fixed transit time on arcs is a collection of commodities defined by $\Phi^i: A^a \times T \rightarrow R^+$, for each arc $e \in A$ such that $\Phi^i(\sigma) = 0$, for $\sigma \geq T - \tau_e$. We define the excess of node x induced by Φ^i at time σ is

$$exc^i_\Phi(x,\sigma) = \sum_{\theta=0}^{\sigma} \sum_{e \in \delta^+(x)} \Phi^i_e(\theta) - \sum_{\theta=\tau_e}^{\sigma} \sum_{e \in \delta^-(x)} \Phi^i_e(\theta - \tau_e), \forall x \in V$$

satisfying the constraints (2 - 4).

$$\min T \tag{1}$$

subject to,

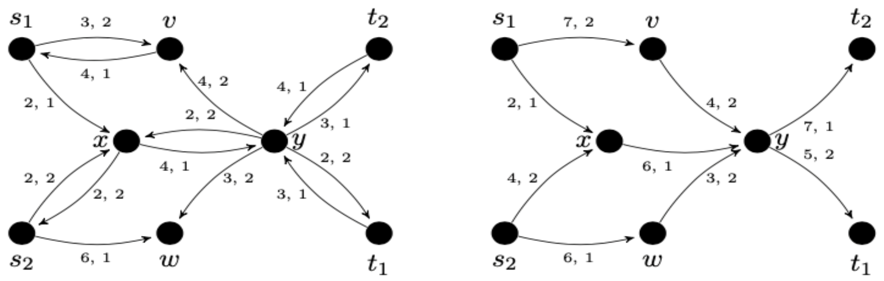
$$exc^i_\Phi(x,T) = \begin{cases} b_i & if\ x = s_i \\ -b_i & if\ x = d_i \\ 0 & \text{otherwise,} \end{cases} \quad \forall\, i \in K \tag{2}$$

$$exc^i_\Phi(x,\sigma) \leq 0\ \forall\, x \notin \{s_i, t_i\}, i \in K, \sigma \in T, \tag{3}$$

$$0 \leq \Phi_e(\sigma) = \sum_{i \in K} \Phi_e^i(\sigma) \leq u_e + u_{e^r}, \forall e \in A^a, i \in K, \sigma \in T. \tag{4}$$

Flow conservation constraints at time horizon T are the last condition of the constraints in (2), whereas constraints in (3) represent non-conservation of flow at intermediate time points $\sigma \in T$. The capacities with lane reversals also limit the bundle limitations in (4). The purpose is to move a certain quantity of flow to meet the demand b_i of each commodity i from s_i to t_i, as indicated in the first two conditions of the equation (2). The strict inequality in (3) denotes modest flow conservation restrictions that enable the flow to be stored at intermediate nodes, waited for a short time (storage is permitted), and then continued ahead. Flow conservation at intermediate nodes is represented by the flow over time obeying the equality condition in (3).

Example 1. Consider the asymmetric capacity and transit time on some anti-parallel arcs of a two-commodity network as indicated in Figure 1. Commodity-1's flows must be transshipped from s_1 to t_1, whereas Commodity-2's flows must be transshipped from s_2 to t_2. Figure 1(b) shows the auxiliary network of Figure 1(a). We split the two-commodity flow problem as two single-commodity flow problems, redefining capacity for every arc per commodity that meets the arc capacity limit.



(a) capacity, transit time

(b) capacity, transit time

Figure 1: (a) Represents dynamic multi-commodity network (b) the auxiliary network of (a).

The quickest contraflow problem for a single commodity is described as an integer programming problem (Kim et al., 2008). For the numerical solution, they also developed greedy and bottleneck relief strategies. It can be solved polynomially in its single-source and single-sink variant (Rebennack et al., 2010). However, as it is equivalent to 3-SAT and PARTITION, the issue with numerous sources and/or sinks is NP-hard. Pyakurel and Dhamala (2017) and Pyakurel et al. (2017) have shown that how to solve the continuous-time variant in polynomial-time. The problem with partial lane reversals is investigated in Pyakurel et al. (2019).

APPROXIMATION SCHEME

Consider problem X to be an optimization problem. For example, let $OPT(I)$ be the optimum solution of the objective function, and let $\epsilon > 0$. For each instance I of X, an algorithm A is considered a $(1+\epsilon)$ (or $(1-\epsilon)$) approximation algorithm if it yields a feasible solution with objective value $A(I)$ such that $|A(I) - OPT(I)| \leq \epsilon OPT(I)$.

An approximation scheme is a polynomial-time approximation scheme if its computational time depends on input size of the problem, whereas it becomes FPTAS if its computational time is polynomial in input size as well as $1/\epsilon$.

In many cases, it is better to generate an approximate solution rapidly rather than an optimum one. As a result, a concerted effort was undertaken to develop effective FPTAS for the multi-commodity flow problem.

QMCCF WITH ASYMMETRIC TRANSIT TIMES

This section discusses the approximation approach for solving the QMCCF problem with asymmetric transit times on anti-parallel arcs. In a network where arc reversals are permitted, a solution to this problem satisfies specific demands at specific nodes in the quickest time possible. We also devise a method for getting an approximate solution to this problem that is both efficient and effective. Our method may also be used to save lanes that don't need to be reversed to save time. This method extends the network flow models introduced in the lane reversal framework introduced in Dhamala et al. (2018) and Pyakurel et al. (2019) in the case of asymmetric transit times on anti-parallel arcs.

Problem 1. Consider the following network with asymmetric transit times on arcs $G = (V, A, K, u, \tau, b_i, S_+, S_-, T)$. The QMCCF problem aims to compute the minimum feasible time that is needed to transship a given number of commodities b_i from initial points to the corresponding terminal points for each commodity by flipping the orientation of arcs required at time zero and satisfying the criteria (2), (3), and (4).

Even with series-parallel networks dynamic multi-commodity flow problem having only two commodities, with or without intermediate node storage is NP-hard (Hall et al., 2007). The proof uses reductions from the NP-hard PARTITION and 3-PARTITION problems. The NP-hardness of the maximum dynamic multi-commodity flow is also a concern. By using modified time-expanded network this problem can be solved as a static flow problem in a pseudo-polynomial time and no storage limitations on intermediate nodes. According to Hall et al. (2007), the quickest multi-commodity flow without intermediate node holding and the simplest flow path are complicated. Using 3-SAT, Kim et al. (2008) have shown that the lane reversal problem is NP-complete. As a result, We have the following Theorem 1.

Theorem 1. The QMCCF problem with asymmetric transit times on anti-parallel arcs is NP-hard.

Due to its NP-hardness, Fleischer and Skutella (2007) proposed two techniques to provide an approximate solution to the quickest multi-commodity flow problem. The first is length-bounded flow, while the second involves discretization of bigger instead of unit time steps. Dhamala et al. (2020) and Gupta et al. (2020) introduced the partial lane reversal approach in the quickest multi-commodity flow problem and provided the approximate solution in both cases. We extend the approach of Dhamala et al. (2020) and Gupta et al. (2020) in the case of non-symmetric transit times on anti-parallel arcs and provide the solution of the problem by using a Δ-condensed time-expanded network.

AN FPTAS FOR THE QMCCF PROBLEM WITH ASYMMETRIC TRANSIT TIMES

For multi-commodity flows, consider a network $G = (V, A, K, u, \tau, b_i, S_+, S_-, T)$, assuming that parameters are integers. By rescaling the transit time as defined above, we get a fully polynomial-time solution. According to Fleischer and Skutella (2002, 2007) by rescaling the time, the condensed time-expanded network may be defined as $G_T^\Delta = (V_T^\Delta, A_M^\Delta \cup A_H^\Delta)$, assuming that arc transit times are multiple of $\Delta > 0$, where the sets of vertices and arcs are specified as

$$V_T^\Delta = \left\{x_{\alpha\Delta} : x \in V, \alpha = 0, 1, 2, \dots, \left\lceil \frac{T}{\Delta} \right\rceil \right\}$$

$$A_M^\Delta = \{(x_{\alpha\Delta}, y_{\alpha\Delta+\tau_e}) : e = (x, y) \in A, \alpha = 0, 1, \dots, \lceil (T - \tau_e)\Delta \rceil\}$$

$$A_H^\Delta = \{(x_{\alpha\Delta}, x_{\alpha\Delta+1}) : e = (x, y) \in A, \alpha = 0, 1, \dots, \lceil T/\Delta \rceil - 1\}$$

The replicas of V_T^Δ correspond to commodity flow via V in time $T = \{\alpha\Delta\}$ or $[\alpha\Delta, (\alpha + 1)\Delta)$ for discrete-time or continuous-time, wherein $\alpha = \{0, 1, 2, \dots, \lceil T/\Delta \rceil\}$. In this configuration, capacities are rescaled by Δu_e for each arc corresponding to a discrete-time with multiple of Δ. Any dynamic multi-commodity that accomplishes by time T corresponds to a static multi-commodity flow of similar value in G_T^Δ (Fleischer & Skutella, 2007; Gupta et al., 2022), while $\frac{T}{\Delta}$ is an integral and arc length is defined as before. In the same way, each flow in G_T^Δ corresponds to an equal-valued flow over time that completes by time T. This network transforms to the standard time-expanded network, if we assume $\Delta = 1$. When arc travel times are not multiple of Δ, then they are rounded up to the nearest multiple of Δ by $\tau_e' = \left\lceil \frac{\tau_e}{\Delta} \right\rceil \Delta$, $0 \leq \tau_e' - \tau_e < \Delta$ for all arcs $e \in A$, then we have, $0 \leq \tau_p' - \tau_p < n\Delta = \epsilon^2 T$.

A simple concept is to decrease the size of the time-expanded network by substituting unit- length time steps with larger ones. A condensed time-expanded network of polynomial-size results from a suitably rough discretization of time. Furthermore, there is a tradeoff between the need to shrink the time-expanded network and the aim to keep the resultant flow model as precise as possible because the latter leads to a reduction in the quality of feasible solutions. This tradeoff can be solved in a suitable manner. For every $\epsilon > 0$, an acceptable choice of step length results in a condensed time-expanded network of the polynomial-size that nevertheless permits a $(1 + \epsilon)$-approximate accuracy in time.

In theory, we may choose a Δ and round all travel times to the nearest multiple of Δ. This introduces a rounding error, which results in two issues: increased path length and path length distortion.

• route lengths should increase by no more than $1 + \epsilon$ for $\epsilon > 0$.
• and the number of time layers after scaling should be polynomial in the original network's size, with $\epsilon^{-1}: T/\Delta \in O(poly(n, \epsilon^{-1}))$.

We will go with $\Delta = \frac{\epsilon^2 T}{n}$ for now. Because the maximum rounding for a path is $\epsilon^2 T$ and the number of time layers $n\epsilon^{-2}$, this meets both conditions.

We describe Algorithm 1 to provide the solution to Problem 1 by employing a Δ-condensed time-expanded network. We build a Δ-condensed auxiliary network in which arc capacities are Δ times the sum of the capacities of forward and backward arcs of the provided network. Commodities are shipped via transformed network G provides the solution to QMCCF problem with asymmetric transit times using FPTAS of Fleischer and Skutella (2007) and the lane reversal approach of Gupta et al. (2021a).

Algorithm 1. An FPTAS for QMCCF problem with asymmetric transit times.

Input : Consider multi-commodity network $G = (V, A, K, u, \tau, b_i, S_+, S_-, T)$, with asymmetric transit times on anti-parallel arcs

Output: The quickest multi-commodity contraflow

1. The network G^a is converted to Δ-condensed auxiliary network
$G^{\Delta a} = (V^{\Delta a}, A^{\Delta a}, K, u', \tau', b, S'_+, S'_-, T)$ with

$$u'_a = \Delta(u_e + u_{e^r})$$

$$\tau'_a = \begin{cases} \left\lceil \frac{\tau_e}{\Delta} \right\rceil \Delta, \text{if arc } e^r \text{ reversed towards } e \\ \left\lceil \frac{\tau_{e^r}}{\Delta} \right\rceil \Delta, \text{if arc e reversed towards } e^r. \end{cases}$$

2. Compute the quickest multi-commodity flow on $G^{\Delta a}$ by using Fleischer and Skutella (2007) with intermediate node storage.
3. Remove the flows in cycles, $\forall i$ by decomposing the flow along the $s_i - t_i$ pathways and cycles.
4. Reverse $e^r \in A$ up to the arc capacity $f_e - u_e$ if and only if $f_e > u_e$, $u_e$replaced by 0, whenever $e \notin A, \forall i$,where $f_e = \sum_{i=1}^{k} f_e^i$ and $u_e = \sum_{i=1}^{k} u_e^i$.
5. For each $e \in A$, if e is reversed, $s_c(e) = u_a - f_e$ and saved capacity of arc e is zero. If neither forward nor backward arc is reversed, the saved capacity of the forward arc is $u_e - f_e > 0$.

In the first step of the algorithm Δ-condensed auxiliary network $G^{\Delta a}$ is constructed. This transformation allows us to reduce the QMCCF problem to the quickest multi-commodity flow problem on the transformed graph in Step 2. Hence, we can calculate the quickest multi-commodity flow according to Fleischer and Skutella (2007). Step 3 removes the cycle flows on the transformed network, so the flow moves in only one direction, but not both. Thus, Step 4 of the algorithm is well defined. Step 5 saves the unused capacity of the arcs. Hence all the steps of the algorithm are feasible. As a consequence, we have Lemma 1.

Lemma 1. The solution of the QMCCF with non-symmetric transit times obtained by Algorithm 1 is feasible.

Theorem 2. An FPTAS provides an approximate solution to the QMCCF problem with non-symmetric transit times by using Algorithm 1.

Proof: The theorem will be proved in two steps feasibility and optimality. Lemma 1 proves the feasibility of the algorithm. In the next step, we prove optimality.

On the converted network $G^{\Delta a}$, an optimum solution to Problem 1 on network G is likewise a feasible solution to the approximate QMCF. By reducing dynamic multi-commodity flow to a static flow problem, the pseudo-polynomial time solution, on time-expanded networks is produced (Skutella, 2009). The reduction of the network size by a factor of Δ yields an approximate polynomial-time bound.

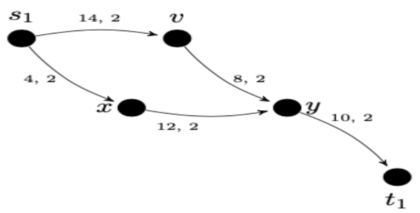
An estimated quickest flow solution can be optimally computed on network G. Multi-commodity flow problems can be reduced to single-commodity flow problems by sending flow s_i to $t_i, \forall i \in K$. Furthermore, any optimum solution on $G^{\Delta a}$ is the same as the possible solution to the specified network G. Thus, on network G, an approximate QMCCF solution with non-symmetric transit times can be calculated optimally.

Corollary 1. An approximate solution to the QMCCF problem with non-symmetric transit times can be computed in fully polynomial-time complexity.

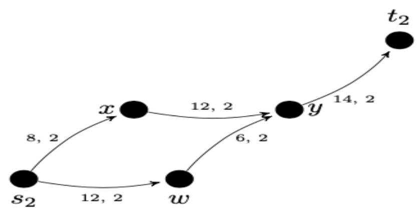
Proof: The complexity of Algorithm 1 is dominated by Steps 2 and 3. The remaining steps can be accomplished in $O(m)$ times. Step 3 is executed in $O(mn)$ times. A $(1 + \epsilon)$ approximate solution of static multi-commodity flow problem can be computed by $O(\log 1/\epsilon)$ computation in a G_T^Δ , [7]. There are $(\frac{n}{\epsilon^2})$ layers with vertices $(\frac{n^2}{\epsilon^2})$ and arcs $(\frac{mn}{\epsilon^2})$ in G_T^Δ. The complexity of the algorithm depends on input size as well as $1/\epsilon$. According to [20], the complexity of lane reversal problems with non-symmetric transit time is the same as symmetric transit times on anti-parallel arcs. As a result, the solution can be obtained in fully polynomial-time.

Example 2. Consider the two-commodity network from Figure (1)(a) having demands $b_1 = 10$ and $b_2 = 12$. Figure 1(a) is used to calculate the quickest time without lane reversals by using Δ-condensed time-expanded network taking $\Delta = 2$, and rescaling the capacity and transit times, respectively. We have only one path for commodity-1, i.e., $s_1 - x - y - t_1$ having transit time 6, and flow along the path is 4. Similarly, commodity-2 has path $s_2 - x - y - t_2$ having transit time 6, and flow along the path is 4. The minimum time to fulfill both the demands is $T = 10$.

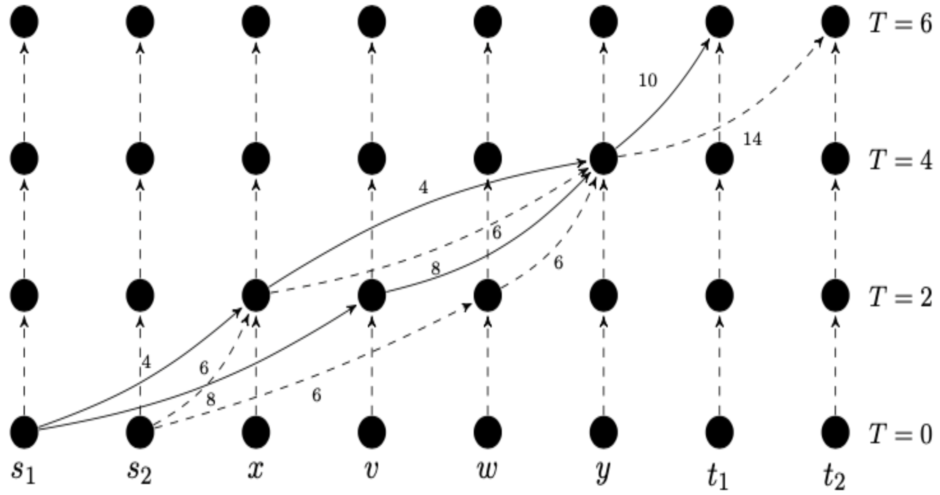
Although, if we flip the orientation of arcs, i.e., with lane reversals (c.f. Figure 1(b)) and rescale the capacity and transit times on arcs, which takes $T = 6$ units of time to satisfy both the demands (c.f. Figure 2(a), (b)). So, approximately 40% of the time is saved due to lane reversal.



(a) condensed capacity, transit times



(b) condensed capacity transit times



(c) Δ-condensed time-expanded graph, with Δ = 2.

Figure 2: (a) and (b) represent condensed network after contraflow for Commodity-1 and 2, respectively. (c) condensed time-expanded network for Commodity-1 and 2.

The outcomes of Example 2 are summarized as follows.

Table 1: Quickest time without and with lane reversals by using Δ-condensed network

Δ-condensed without lane reversals	Δ-condensed with lane reversals
10	6

CONCLUSIONS

One of the fundamental difficulties in operations research is routing many commodities from their origin to their destination over a shared network. The reduction of time (cost) is critical. A well-known quickest flow problem was explored to meet the demand in the shortest feasible period. The quickest flow problem can be solved in polynomial-time in the single-commodity situation, but NP-hardness exists for the multi-commodity case. However, a polynomial-time approximation technique based on a length bound function as well as an FPTAS based on a Δ-condensed time-extended network have been developed. Lane reversals technique is a significant tool for improving the quickest time in a two-way network. This approach is used for both length-bounded approximation and condensed time-expanded networks.

The quickest multi-commodity contraflow problem with asymmetric transit time over anti-parallel arcs was explored in this work. We introduce its mathematical model and provide an FPTAS to solve the problem. We are interested in extending these techniques to flow-dependent attributes, as we have examined the problem with constant transit time. The findings of this research are both theoretical and practical in nature. Researchers that wanted to expand their concepts for time-dependent, flow-dependent, and load-dependent attributes would find this study useful.

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AUTHOR CONTRIBUTIONS

SPG: Conceptualization, investigation, documentation, and editing; TND: Supervision.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest regarding the publication of this paper.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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