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***Short note***

# THE SOLVABILITY OF POLYNOMIAL PELL'S EQUATION

**Bal Bahadur Tamang**[1,2*], **Ajay Singh**[1]

[1]*Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal*
[2]*Department of Mathematics, M. R. M. Campus, Tribhuvan University, Ilam, Nepal*
[*]*Correspondence author: tamang.bb100@gmail.com*

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## ABSTRACT

This article attempts to describe the continued fraction expansion of $\sqrt{D}$ viewed as a Laurent series $x^{-1}$. As the behavior of the continued fraction expansion of $\sqrt{D}$ is related to the solvability of the polynomial Pell's equation $p^2 - Dq^2 = 1$ where, $D = f^2 + 2g$ is monic quadratic polynomial with $deg\ g < deg\ f$ and the solutions $p, q$ must be integer polynomials. It gives a non-trivial solution if and only if the continued fraction expansion of $\sqrt{D}$ is periodic.

**Keywords:** Continued fraction, Diophantine equation, Integers, Polynomial Pell's equation.

## INTRODUCTION

Number theory is a collection of areas of pure Mathematics. The objective of number theory is the study of integers. The theory of Pell's equation has a long history as can be seen from the huge amount of references collected in Dickson (1950) from the two books on its history by Konen (1901) and Whitford (1912). So, Pell's equation is studied in number theory.

Diophantine equation of the form

$$x^2 - dy^2 = 1 \tag{1}$$

Where, $d$ is a positive integer, not perfect square, is known as the classical Pell's equation (Niven *et al.*, 1991). Geometrically, the set of integer solutions $(x, y)$ is the set of intersections of a hyperbola with the lattice in integers. Integer solutions of the equation (1) were well understood by the contributions of Euler, Lagrange and others.

Pell's equation was studied by Brahmagupta (598-670) and Bhaskara (1114-1185) in Arya (1991). It is often said that Euler (1707-1783) mistakenly attributed Brounckers (1620-1684) work on this equation to Pell. The original algorithm is for solving Pell's equation after Euclid's algorithm.

Let $(x, y)$ be a solution to equation (1), then

$$(x - y\sqrt{d})(x + y\sqrt{d}) = x^2 - dy^2 = 1$$

$$\Rightarrow \left|\sqrt{d} - \frac{x}{y}\right| = \frac{1}{y^2(\sqrt{d} + \frac{x}{y})} < \frac{1}{2y^2}$$

Hence, x/y is the best approximation to irrational number $\sqrt{d}$ in Burton (1980). It follows that all solutions of the equation (1) can be found among the convergent to $\sqrt{d}$. Let $r$ be the length of the period of expansion $\sqrt{d}$.

If r is odd, then all positive solutions are $(x, y) = (p_{2kr-1}, q_{2kr-1})$. If r is even, then $(x, y) = (p_{kr-1}, q_{kr-1})$ where $k = 1, 2, \cdots$ and $p_n$, $q_n$ is $n^{th}$ convergent of the continued fraction expansion of $\sqrt{d}$ in Kumundury and Romero (1998).

Lagrange (1768) was first to prove that the equation (1) has infinitely many solutions and it gives a non-trivial solution in Niven *et al.* (1991).

**Theorem 1** (Niven *et al.*, 1991)

If $x_1, y_1$ is the fundamental solution of Pell's equation; $x^2 - dy^2 = 1$, where $d$ is a positive integer, not a perfect square. Then, all positive solutions are given by $x_n, y_n$ for $n = 1, 2, \cdots$ where $x_n$ and $y_n$ are the integers defined by $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$. So, the value of $x_n$ and $y_n$ are determined by expanding the power and equating the rational parts and the purely irrational parts. The first solution $x_1, y_1$ is called the fundamental solution to Pell's equation and solving the Pell's equation means finding the value of $x_1, y_1$ for a given d. Also, Tekcan (2011) provided a formula for the continued fraction expansion of $\sqrt{d}$ for some specific values of $d$ with $d \neq 1$, then considered the integer solutions of the equation (1). So, we consider the continued fraction expansion $\sqrt{d}$ can be defined in many ways depending on the base field (Mollin (1997; Ramasamy, 1994).

## Preliminaries

### Sign function

Let $x$ be variable. Then the sign function is denoted by $sgn(x)$ and defined by

$$sgn(x) = \begin{cases} 1, if\ x > 0 \\ 0, if\ x = 0 \\ -1, if\ x < 0 \end{cases}$$

Thus, sign function takes a value and returns whether that value is positive, negative and zero.

### Polynomials over a field

Let $F$ be a field. A polynomial over $F$ is

$$f(x) = \sum_{r=0}^{n} a_r x^r = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Where, $a_0, a_1, a_2, \dots, a_n \in F$ and $x$ is indeterminate in Nagell (1951). Let $F[x]$ denote the set of all polynomials over $F$. The degree of a polynomial $f$ is the largest power of $x$ whose coefficients $f(x)$ are non-zero. It is denoted by $\deg f$. In this case, $a_n x^n$ is called the leading term $f(x)$ and $a_n$ is leading coefficient. A polynomial is monic if its leading coefficients are equal to one.

### Periodic

An infinite sequence $(a_n)_{n\geq 1}$ is periodic if there exists a positive integer $s$ such that $a_{n+s} = a$ for all $n \geq 1$. In this case, the finite sequence $(a_1,a_2,\dots,a_s)$ is called a period of the original sequence. It is denoted by $(a_1, a_2, \dots) = (\overline{a_1, a_2, \dots, a_s})$

### Polynomial Pell's equation

Diophantine equation is a polynomial equation with two or more unknowns in which only integer solutions are studied. The Diophantine problems consist of unknown variables involved in finding the integer solutions that work correctly for all the equations.

Diophantine equation of the form

$$p^2 - Dq^2 = 1 \qquad (2)$$

Where, $p$ and $q$ are polynomial with integer coefficients and $D$ is monic quadratic polynomial with integer coefficients is known as polynomial Pell's equation.

Clearly, if $\deg D = 0$, then $\deg p = \deg q = 0$. Since the set of solutions of Pell's equation in integers is well understood, we may assume in the sequel that $\deg D > 0$. The polynomial Pell's equation has no solutions if $\deg D$ is an odd number. Therefore, we assume that $\deg D$ is an even number, so that $\deg D \geq 2$. Also, if $(p,q)$ is a non-trivial solution, then so are $(p,-q)$ and $(-p,q)$. Sometimes the expression $(p + q\sqrt{D})$ is called a solution of equation (2), where $(p,q)$ is also a solution of equation (2) in Dubickas and Steuding (2004). Given $D = f^2 + 2g$ is monic quadratic polynomial with $\deg g < \deg f$, it is known that the equation (2) is solvable in $\mathbb{Q}[x]$ if and only if the periodic of the continued fraction of $\sqrt{D}$ is an even degree in Malyshev (2004).

### Continued fraction expression of $\sqrt{D}$ define on the base field

The general theory of Pell's equation based on continued fraction and algebraic manipulations with the form $p + q\sqrt{D}$ was developed by Lagrange (1766-1769) in Serret (1867). Today, continued fractions of real numbers remain an important research topic in number theory and other branches of mathematics.

We write a continued fraction as

$$\xi = [a_0, a_1, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

For the classical continued fractions with $\xi \in \mathbb{R}$, the partial quotients $a_n$ are integers, positive for $n > 0$ in Olds (1963). Instead, one may also take $a_n \in \mathbb{Q}[x]$ to be polynomials, non-constant for $n > 0$ to build the continued fraction of a Laurent series in $x^{-1}$. The role of the nearest integer is then played by the polynomial part of the Laurent series. We now explain how to compute $\sqrt{D}$ as a Laurent series in $x^{-1}$.

Let F be an arbitrary field and $\kappa = F((x^{-1}))$ be the field of Laurent series in $x^{-1}$ over $F$. This is an extension field of in $F[x]$, the field of rational function of $x$. So, the usual theory of continued fraction carries over $K$, with the polynomials in $x$ playing the role of the integers.

Let $\kappa = \mathbb{Q}((x^{-1}))$ be the field of Laurent series in $x^{-1}$ over $\mathbb{Q}$.

Then $\alpha \in \kappa \Rightarrow \alpha = \sum_{j=t}^{\infty} a_j x^{-j}$, where $a_j \in \mathbb{Q}$, for all $j \in \mathbb{Z}$ and $t \in \mathbb{Z}$ such that $a_t \neq 0, sgn\alpha = a_t$ (Webb, 2006)

The degree evaluation $V_\infty(\alpha)$ of $\alpha$ is $-t$, and absolute value $|\alpha|_\infty$ of $\alpha$ is $e^{-t}$. So, we define the non-Archimedean absolute value by $|\alpha| = e^{-t}$

Thus, $\left|\frac{f}{g}\right| = e^{\deg f - \deg g}$, for $f, g \in \mathbb{Q}[x]$

Thus, $\lfloor\alpha\rfloor = \sum_{n=t}^{0} a_t x^{-t} = a_t x^{-t} + \cdots + a_0 \in \mathbb{Q}[x]$, where, $\lfloor\alpha\rfloor$ is integer part of $\alpha$, for the integral part or polynomial part of $\sqrt{D}$ was used by Artin (1924), and Baum and Sweet (1976) for their continued fraction. We construct a continued fraction expansion of a Laurent series in Baum and Sweet (1976) as follows;

For, $D \in \mathbb{Z}[x]$, a continued fraction for $\sqrt{D}$ is obtained by putting $\alpha_0 = \sqrt{D}$ and recursively for $n \geq 0$. Putting $A_n = \lfloor\alpha_n\rfloor$, $\alpha_{n+1} = \frac{1}{\alpha_n - A_n}$. The algorithms terminates, if for some $n$, $\alpha_n = A_n$. This happens if and only if $\sqrt{D}$ is a rational function. Then

$$\sqrt{D} = \lfloor\sqrt{D}\rfloor + \frac{1}{\alpha_1}$$

$$= \lfloor\sqrt{D}\rfloor + \frac{1}{\lfloor\alpha_1\rfloor + \frac{1}{\alpha_2 + \ddots}}$$

$$= [\lfloor\sqrt{D}\rfloor, \lfloor\alpha_1\rfloor, \ldots]$$

$$= [A_0, A_1, \ldots], where\ A_i \in \mathbb{Q}[x].$$

So, we write convergent to $\sqrt{D}$ as $\frac{p_n}{q_n} = [A_0, A_1, \ldots]$ where

$$\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} A_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{n-1} & q_{n-1} \\ p_{n-2} & q_{n-2} \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} p_{-1} & q_{-1} \\ p_{-2} & q_{-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the determinant of the given matrix

$$\begin{vmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{vmatrix} = p_n q_{n-1} - q_n p_{n-1}$$

$$= (-s1)^{n+1}, \text{ for } n \geq 0.$$

Since, $sgnA_n > 0, \sigma(p_n) = \sigma(q_n)$, for all $n \geq 0$, where $\sigma(A)$ is the sign of the leading coefficient of $A$.

Thus, $\sqrt{D} = [A_0, A_1, \ldots, A_n, A_{n+1}, \ldots]$

$$= [A_0, A_1, \ldots, A_n, \alpha_{n+1}]$$

Hence ,

$$\sqrt{D} = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}$$

## MATERIALS AND METHODS

In general Pell's equation (1) always has non-trivial solutions $x, y$ when $d$ is a positive integer, not a perfect square. The major objective is to determine the polynomial $D$ for which equation (2) has non-trivial solutions in $D[x]$ where $D = f^2 + 2g$ is monic quadratic polynomial with $\deg g < \deg f$. It is a descriptive study where the proposition is proved through theorem and examples by using number theoretic approach. The main result of the polynomial Pell's equation was based on a review and discussion of the previously published documents.

## RESULTS

### Solvability of Polynomial Pell's equation

We begin by exploring some well-known basic properties of the Pell's equation over polynomials, usually called the polynomial Pell's equation. We also explain how to write square roots of polynomials in $x$ as Laurent series $x^{-1}$ and use this to show that the group of solutions of the polynomial Pell's equation has ranked at most one. It was first considered to study the integration in elementary terms of certain algebraic functions (Abel, 1826). He showed that the periodicity of the continued fraction is equivalent to the existence of a non-trivial solution $(p, q) \in \mathbb{Q}[x]$, $q \neq 0$ of the polynomial Pell's equation (2). Also, it was shown that $D$ is Pellian if and only if continued fraction expansion of $\sqrt{D}$ is periodic and Pell's equation has a non-trivial solution (Abel, 1826). Solving the Pell's equation in $\mathbb{Z}[x]$ has been studied by Mollin (1997). Dubickas and Steuding (2004) reported the polynomial solutions of the equation (2). The solutions $(p, q) = (\pm 1, 0)$ are trivial solutions. All other solutions are non-trivial. The main difficulty in solving polynomial Pell's equations is to determine whether non-trivial solutions exist or not.

Chowla (1982) asked for the solutions of the equation (2) in $\mathbb{Z}[x]$ for $D = x^2 + k \in \mathbb{Z}[x]$. Nathanson (1976) proved that there are no non-trivial solutions of the equation (2) when, $k \neq \pm 1, \pm 2$. If, $k = 1, \pm 2$, then there are non-trivial solution of the equation (2) and he found that the sequences of polynomial given by

$$p_n = \left(\frac{2x^2}{k} + 1\right) P_{n-1} + \frac{2x}{k}(x^2 + k) Q_{n-1}$$

$$q_n = \frac{2x}{k} P_{n-1} + \left(\frac{2x^2}{k} + 1\right) Q_{n-1}$$

Where, $p_0 = 1, q_0 = 0$, for all $n \in \mathbb{N}$ and showed that the only integer polynomials which satisfy the equation (2) are the form $(\pm \mathrm{p_n}, \pm \mathrm{q_n})$ and for $k = -1$, he gave another family of solutions in Nathanson (1976). These polynomials can be expressed as Chebyshev polynomials in Pastor (2001). Gaunct (1990) proved a similar result for a cubic analog of equation (2). Hazama (1997) studied the polynomial Pell's equation using the twist of a conic by another conic. Webb and Yokota (2003) found that the necessary and sufficient condition for which the equation (2) has a non-trivial solution when $D = f^2 + 2g$ is monic polynomial, where $f, g \in \mathbb{Z}[x]$ and $\frac{f}{g} \in \mathbb{Z}[x]$.

Such result is generalized when $p \frac{f}{g} \in \mathbb{Z}[x]$ in Webb and Yokota (2004), where $p$ is prime without any condition of $\deg g$. In this case, the authors also determined the solutions. Then, Yokota (2010) found that a necessary and sufficient condition for the solution of the polynomial Pell's equation, when $\frac{f}{g} \in \mathbb{Q}[x]$. Langhlin (2018) focused on the relation between polynomial solutions of Pell's equation and fundamental units of real quadratic fields. Zapponi (2016) studied polynomial solution of the equation (2) in $\mathbb{C}[x]$. If $D$ is a perfect square, then equation (2) has no non-trivial solution. For

$$p^2 - Dq^2 = 1$$

$$\Rightarrow (p + q\sqrt{D})(p - q\sqrt{D}) = 1$$

$$\Rightarrow p = \pm 1, q = 0$$

$$\Rightarrow (p, q) = (\pm 1, 0)$$

**Example:** A trivial solution of an equation $x^2 - 5y^2 = 1$ is $(x, y) = (\pm 1, 0)$

**Example:** A non-trivial solution of an equation $x^2 - 5y^2 = 1$ is $(x, y) = (\pm 9, \pm 4)$

Let $D$ be a monic quadratic polynomial in $\mathbb{Z}[X]$. Suppose that the period of the continued fraction expansion of $\sqrt{D}$. Then the polynomial Pell's equation (2) has no non-trivial solutions $p, q \in \mathbb{Z}[X]$ (Yokota, 2010).

Similarly, for large value of *D*, Polynomial Pell's equation (2) may obviously have small integer solutions in Waldschmidt (2016).

**Example:** For $D = m^2 - 1$ with $m \geq 2$, the number $p = m$, $q = 1$ satisfy the equation (2)

**Example:** For $D = m^2 \pm m$ with $m \geq 2$,

the number $p = 2m \pm 1$, $q = 2$ satisfy the equation (2)

**Example:** For $D = t^2m^2 \pm 2m$ with $m \geq 1$ and $t \geq 1$, the number $p = t^2m$, $q = t$ satisfy the equation (2)

On the other hand, relatively small value of *D* may leads to large fundamental solutions.

**Theorem 2** (Webb & Yokota, 2002)

Let $D = f^2 + 2g$ be a monic polynomial in $\mathbb{Z}[x]$, where $\deg g < \deg f$. Suppose that

$\sqrt{D} = [A_0, A_1, \dots, A_n, \alpha_{n+1}]$. Then $\alpha_{n+1}$ is reduced, $\deg A_n \geq 1$ and

$\left|\frac{p_n}{q_n} - \sqrt{D}\right| = \left|\frac{1}{q_n q_{n+1}}\right|$, for all $n \geq 0$.

Proof

We want to show that by using induction on $n$ that $\alpha_n$ is reduced, and $\deg n \geq 1$, for all $n > 0$. Since, $D = f^2 + 2g$ with $\deg g < \deg f$ and $\lfloor \sqrt{f^2 + 2g} \rfloor = f$. If $\sqrt{D} = [A_0, A_1, \dots A_n, \alpha_{n+1}]$, then $A_0 = f$ and $\deg A_0 \geq 1$. Since, $D$ is monic, $sgn\ f > 0$ and $|\sqrt{D} + f| = e^{\deg f}$, then $|\sqrt{D} - f| = \left|\frac{D - f^2}{\sqrt{D} + f}\right|$

$$= \left|\frac{2g}{\sqrt{D} + f}\right| < 1$$

So, $|\alpha_1| = \left|\frac{1}{\sqrt{D} - f}\right| = \left|\frac{\sqrt{D} + f}{2g}\right| > 1$ and $|\overline{\alpha_1}| = \left|\frac{1}{\sqrt{D} + f}\right| < 1$.

This shows that $\alpha_1$ is reduced and $\deg A_1 = \deg \lfloor \alpha_1 \rfloor \geq 1$

Suppose $|\alpha_1| > 1$, $|\overline{\alpha_1}| < 1$ and $\deg A_k \geq 1$. Since $|\alpha_k - A_k| = |\alpha_k - \lfloor \alpha_k \rfloor| < 1$, then we have

$|\alpha_{k+1}| = \left|\frac{1}{\alpha_k - A_k}\right| > 1$, Since. $|\overline{\alpha_1} - A_k| = |\alpha_K| > 1$, then we have, $|\overline{\alpha_k}| = \left|\frac{1}{\overline{\alpha_k} - A_k}\right| < 1$.

Hence, $\alpha_{k+1}$ is reduced and $\deg A_k = \deg \alpha_k \geq 1$.

Next, we want to show that $\left|\frac{p_n}{q_n} - \sqrt{D}\right| = \left|\frac{1}{q_n q_{n+1}}\right|$, for all $n \geq 0$. We assume that $|\alpha_{n+2}| > 1$, since $\left|\frac{q_n^2}{\alpha_{n+2}}\right| < |q_n q_{n+1}|$

Then, we have

$$\left|\frac{p_n}{q_n} - \sqrt{D}\right| = \left|\frac{p_n}{q_n} - \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n+1}}\right|$$

$$= \left|\frac{p_n q_{n-1} - q_n p_{n-1}}{q_n(\alpha_{n+1}q_n + q_{n-1})}\right|$$

$$= \left|\frac{1}{q_n\left(\left(A_{n+1} + \frac{1}{\alpha_{n+2}}\right)q_n + q_{n-1}\right)}\right|$$

$$= \left|\frac{1}{q_n q_{n+1} + \frac{q_n^2}{\alpha_{n+2}}}\right|$$

$$= \left|\frac{1}{q_n q_{n+1}}\right|$$

Finding polynomial solutions to Pell's equation is of interest as such solutions sometimes allow the fundamental units to be determined in an infinite class of real quadratic fields as described elsewhere (Langhlin, 2018).

## DISCUSSION

The solution of Pell's equation has been applied in many branches of mathematics. Most basically, $\frac{p_k}{q_k}$ approximates $\sqrt{D}$ arbitrarily closely, where $(p_k, q_k)$ is $k^{th}$ the solution for $D$ in Olds (1963). Stormer's theorem applies Pell's equations to find pairs of consecutive smooth numbers and the most significant application of the Pell's equation was done in Matiyasevich (2017). It gives every computably enumerable set is Diophantine. We formalize theorems related to the solvability of Pell's equation imitating the approach considered in Sierpinski (1964) and Dirichlet's approximation theorem to show that $|p - q\sqrt{D}|$ can be arbitrarily close to zero. Then there exist infinitely many pairs $(p, q)$ where $|p^2 - Dq^2| < 2\sqrt{D} + 1$. Suppose $w = u + v\sqrt{D}$ is a rational solution of equation (2), if $u^2 - Dv^2 = 1$ and $u, v \in \mathbb{Q}[x]$

We define,

$$T = \left\{\begin{matrix} u + v\sqrt{D} : u^2 - Dv^2 = 1, sgn\ u > 0, sgn\ v > 0, \\ \text{where } u, v \in \mathbb{Q}[x] \end{matrix}\right\}$$

and $T_0$ is a subset of $T$ such that $u, v \in \mathbb{Z}[x]$. Since $w$ is a rational solution of equation (2). Then $\pm w$ and $\pm \overline{w}$ are solutions of equation (2). Thus, to determine all rational solutions of equation (2), it suffices to all solutions in $T$ in Webb and Yokota (2004). Among all solutions in T, say $p + q\sqrt{D}$ is a fundamental solution if and only if its non-Archimedean absolute value satisfies the condition

$|p + q\sqrt{D}| \leq |u + v\sqrt{D}|$, for all $u + v\sqrt{D} \in T$

We write $\sqrt{D} = \lfloor\sqrt{D}\rfloor + \frac{1}{\alpha_1} = f + \frac{1}{\alpha_1}$. Then, $sgn\ f > 0$.

**Theorem 3** (Webb & Yokota (2002)

If $u + v\sqrt{D} \in T$, then $u = \lambda p_n$ and $v = \lambda q_n$ for some $n \geq 0$ and $\lambda \in \mathbb{Q}$.

Proof

Since

$$\left|\frac{u}{v} - \sqrt{D}\right| = \left|\frac{1}{v(u + v\sqrt{D})}\right|$$

$$= \left|\frac{1}{u^2\left(\frac{u}{v} + \sqrt{D}\right)}\right| < \left|\frac{1}{v}\right|^2$$

We choose $n$, so that $|q_n| \leq |v| < |q_{n+1}|$.

It gives

$\left|\frac{u}{v} - \sqrt{D}\right| < \left|\frac{1}{vq_n}\right|$ and $\left|\frac{p_n}{q_n} - \sqrt{D}\right| < \left|\frac{1}{vq_n}\right|$

If $\frac{u}{v} \neq \frac{p_n}{q_n}$

Then

$$\left|\frac{1}{vq_n}\right| \leq \left|\frac{p_n v - q_n u}{vq_n}\right|$$

$$= \left|\frac{p_n}{q_n} - \frac{u}{v}\right|$$

$$= \left|\frac{p_n}{q_n} - \sqrt{D} - \frac{u}{v} + \sqrt{D}\right|$$

$$\leq max\left\{\left|\frac{p_n}{q_n} - \sqrt{D}\right|, \left|\sqrt{D} - \frac{u}{v}\right|\right\}$$

$< \left|\frac{1}{q_n v}\right|$, which is impossible.

Hence, $\frac{u}{v} = \frac{p_n}{q_n}$

We have, $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$ implies $p_n$ and $q_n$ are relatively prime and $u^2 - Dv^2 = 1$ Implies $u$ and $v$ are also relatively prime.

Thus, $\mathrm{u} = \lambda \mathrm{p_n}$ and $\mathrm{v} = \lambda \mathrm{q_n}$ for some $\mathrm{n} \geq 0$ and $\lambda \in \mathbb{Q}$.

**Theorem 4** (Webb & Yokota, 2002)

If $w_1, w_2 \in T$ and $|w_1| = |w_2|$, then, $w_1 = w_2$., the minimal solution is unique in particular

Proof

Let $\mathrm{w_1} = \mathrm{u_1} + \mathrm{v_1}\sqrt{\mathrm{D}}$ and $\mathrm{w_2} = \mathrm{u_2} + \mathrm{v_2}\sqrt{\mathrm{D}}$.

Then we have $\mathrm{w_1} = \lambda \mathrm{p_m} + \lambda \mathrm{q_m}\sqrt{\mathrm{D}}$ and $\mathrm{w_2} = \mu \mathrm{p_n} + \mu \mathrm{q_n}\sqrt{\mathrm{D}}$ for some $\mathrm{m}, \mathrm{n} \geq 0$ and $\lambda, \mu \in \mathrm{Q}$.

Since, $|\mathrm{w_1}| = |\mathrm{w_2}|$, $\deg \mathrm{p_m} = \deg \mathrm{p_n}$, then we have $\mathrm{m} = \mathrm{n}$, thus, $\lambda^2 (\mathrm{p_m^2} - \mathrm{Dq_m^2}) = \mu^2 (\mathrm{p_n^2} - \mathrm{Dq_n^2})$

Since, $\sqrt{\mathrm{D}}$ is irrational, $\lambda = \pm \mu$ and definition of $\mathrm{T}$.

Then we have $\mathrm{w_1} = \mathrm{w_2}$

In particular, if $\mathrm{w_1}$ and $\mathrm{w_2}$ are minimal solution, then by definition of a minimal solution, we have

$|\mathrm{w_1}| = |\mathrm{w_2}| \Rightarrow \mathrm{w_1} = \mathrm{w_2}$.

**Theorem 5** (Webb & Yokota, 2002)

If $\mathrm{w_0}$ is minimal solution, then for any $\mathrm{w} \in \mathrm{T}$, $\mathrm{w} = \mathrm{w_0^n}$ for $\mathrm{n} \geq 1$.

Proof

If $|\mathrm{w}|$ and $|\mathrm{w_0}|^{\mathrm{n}} = |\mathrm{w_0^n}|$. Then, we have, $\mathrm{w} = \mathrm{w_0^n}$.

Otherwise, we choose $n \neq 1$,

So, $|w_0|^n < |w| < |w_0|^{n+1} \Rightarrow 1 < |\overline{\mathrm{w_0}}^{\mathrm{n}}| < |w_0|$, and $\overline{\mathrm{w_0}}^{\mathrm{n}}\mathrm{w}$ is a solution of equation (2). Since $|\overline{\mathrm{w_0}}^{\mathrm{n}}\mathrm{w}| > 1$, then either $\overline{\mathrm{w_0}}^{\mathrm{n}}\mathrm{w}$ or $-\overline{\mathrm{w_0}}^{\mathrm{n}}\mathrm{w}$ is in $\mathrm{T}$, which is impossible, since $|\overline{\mathrm{w_0}}^{\mathrm{n}}\mathrm{w}| < |\mathrm{w_0}|$.

So, theorem (4) gives a minimal solution is unique and theorem (5) gives every rational solution $w \in T$ can be expressed as $w = w_0^n$ for $n \geq 1$, where $w_0$ is minimal solution. So, to determine the polynomials $D$ for which the polynomial Pell's equation (2) has a non-trivial rational solution, it suffices to find the minimal solution.

Let $w_0$ be the minimal solution. Then we claim that $w_0$ in $T_0$ and $w_0^n \in T_0$ even through $w_0 \notin T_0$.

Since $T_0 \subset T, w \in T_0 \Rightarrow w = w_0^n$ for some $n \geq 1$, where $w_0$ is minimal solution.

Note that for any $u + v\sqrt{D} \in T$. Then $|u + v\sqrt{D}| > 1$ and $|u - v\sqrt{D}| < 1$.

So, we have $|u| + |v\sqrt{D}|$.

If $w_1$ and $w_2$ are rational solutions of the equation (2).

Then $w_1 = u_1 + v_1\sqrt{D}$ and $w_2 = u_2 + v_2\sqrt{D}$

So, $1 = u_1^2 + Dv_1^2 = w_1\overline{w_1} = w_2\overline{w_2} = u_2^2 + Dv_2^2$

Thus, $(w_1 w_2)(\overline{w_1 w_2}) = 1$

Hence $w_1 w_2$ is a rational solution of equation (2).

Also, let us consider $p + q\sqrt{D}$ is a minimal solution. Then the theorem (3) gives

$p + q\sqrt{D} = \lambda(p_n + q_n\sqrt{D})$ for some $\lambda \in Q$.

Suppose $D = f^2 + 2g$ is a polynomial in $\mathbb{Z}[x]$, where $f, g \in \mathbb{Q}[x], \deg f < \deg g$ and

let $h = \frac{f}{g} \in \mathbb{Q}[x]$

Since $\sqrt{D} = \lfloor \sqrt{D} \rfloor + \frac{1}{\alpha_1} = f + \frac{1}{\alpha_1}$

Where,

$$\alpha_1 = \frac{1}{\sqrt{D} - f} = \frac{\sqrt{D} + f}{2g} = \left\lfloor \frac{\sqrt{D} + f}{2g} \right\rfloor + \frac{1}{\alpha_2} = h + \frac{1}{\alpha_2}$$

$$\alpha_2 = \sqrt{D} + f = 2f + \sqrt{D} - f = 2f + \frac{1}{\alpha_1}$$

Hence, $\sqrt{D} = [f, \overline{h, 2f}]$ and

$$\begin{aligned} p_1^2 - Dq_1^2 &= (fh + 1)^2 - Dh^2 \\ &= (fh + 1)^2 - (f^2 + 2g)h^2 \\ &= (fh + 1)^2 - (f^2 g^2 + 2fh) \\ &= 1 \end{aligned}$$

Thus, $\sigma(q_1)(p_1 + Dq_1)$ is a non-trivial rational solution in $T$. Note that this may not be the minimal solution.

For this

$$
\begin{aligned}
(kp_0)^2 - D(kq_0)^2 &= k^2(f^2 - (f^2 + 2g)) \\
&= k^2(-2g) \\
&= 1, \text{ if and only if } 2g = -\frac{1}{k^2}
\end{aligned}
$$

Hence $w_0 = kp_0 - Dq_0$ with $sgn(kp_0) > 0$ is the minimal solution if and only if $2g = -\frac{1}{k^2}$

The classical Pell equation can be generalized in a natural way to higher degrees. Indeed, we can observe that the Pell equation arises considering the unitary elements of the quotient filed $\frac{\mathbb{Q}[x]}{x^2-e}$ where $\mathrm{x}^2 - \mathrm{e}, e \in \mathbb{Z}$ is an irreducible polynomial over $\mathbb{Q}$. Thus, considering the unitary elements of $\frac{\mathbb{Q}[x]}{x^3 - c}$ where $\mathrm{c}$ is not a cube, we get the cubic Pell's equation $\mathrm{x}^3 + \mathrm{cy}^3 + \mathrm{cz}^3 - 3\mathrm{axyz} = 1$ for the unknowns $\mathrm{x, y, z}$ (Murru, 2019). Thus, it is natural generalizing the study of the polynomial Pell's equation to higher degrees.

## CONCLUSION

In solving Pell's equation (1) for various value of $d$, it can be observed that some solutions follow a pattern when $d$ has a certain character and at other times, the solutions for a given $d$ can be quite idiosyncratic. We can evaluate all the convergent of the continued fraction expansion of $\sqrt{D}$ as a Laurent series in $x^{-1}$ leading to the solutions and finding non-trivial solution of the equation (2) that follows a pattern in terms of solving a polynomial version of Pell's equation, where $D, p, q$ are polynomials in one or more variables.

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