Differentiation of Fourier Series via Orthogonal Derivative

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ABSTRACT

It is very known that if the operator $\frac{d}{dx}$ acts on each term into a convergent Fourier Series (FS), then it may result a divergent series. This situation is remedied applying the symmetric derivative to FS, which implies the existence of the important Fejér-Lanczos Factors. In this paper, we show that the orthogonal derivative also leads to these Factors.

Keywords: Orthogonal derivative, least squares method, Fejér-Lanczos σ-factors, differentiation of FS

INTRODUCTION

If on the Fourier series (Lanczos 1966):

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] , \qquad (1)$$

convergent in $[-\pi, \pi]$, we apply the operator $\frac{d}{dx}$ results:

$$\frac{d}{dx} f(x) = \sum_{k=1}^{\infty} k [-a_k \sin(kx) + b_k \cos(kx)] \qquad (2)$$

which it may be divergent (Gibbs 1898, Lanczos 1956). This problem was remedied by Lanczos (1956) with $f'(x)$ defined as a Symmetric derivative (Aull 1967, Washburn 2006):

$$f'(x) \equiv \lim_{n \to \infty} \frac{1}{\frac{2\pi}{n}} \left[f_n \left(x + \frac{\pi}{n} \right) - f_n \left(x - \frac{\pi}{n} \right) \right] \qquad (3)$$

with the partial sums:

$$f_n(x) = g_n(x) + h_n(x) ,$$
$$g_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos(kx) , \ h_n(x) = \sum_{k=1}^{n} b_k \sin(kx),$$
$$\qquad (4)$$

resulting the convergent expression:

$$f'(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \sigma_k \frac{d}{dx} [a_k \cos(kx) + b_k \sin(kx)] , \quad (5)$$

with the Fejér-Lanczos Factors (Lanczos 1956, 1961, Jerri 2000):

$$\sigma_0 = 1 , \ \sigma_k = \frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}} , \ k = 1, \dots, n , \ \sigma_n = 0 . \qquad (6)$$

The set of factors σ_k, for a given *n*, is equivalent to a discrete sampling function.

In (2) and (3) we employ two derivatives, however, also there is the orthogonal derivative (Washburn 2006, Groetsch 1998, Shen 1999, Hicks & Liebrok 2000, Burch *et al.* 2005, López-

Bonilla *et al.* 2010, López-Bonilla *et al.* 2010, Diekema & Koornwinder 2012) obtained by Lanczos (1956),

Cioranescu (1938) and Haslam-Jones (1953), hence it is natural to ask if this ultimate derivative leads to relation (5). The answer is yes, to see the next section.

THE ORTHOGONAL DERIVATIVE

Lanczos (Lanczos 1956, Gordon 2009) used the least squares method of Legendre (1806), Gauss (1809), Laplace (1812) to obtain an integral expression for the derivative of a function, that is, differentiation by integration:

$$F'(x) = \lim_{\epsilon \to 0} \frac{3}{2 \epsilon^3} \int_{-\epsilon}^{\epsilon} t \ F(x + t) \, dt , \qquad (7)$$

which may be applied to Fourier case:

$$\frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t \ g_n(x + t) \, dt \ \overset{(4)}{=} \ \frac{3}{2\epsilon^3} \sum_{k=1}^{n} a_k \int_{-\epsilon}^{\epsilon} t \ \cos(kx + kt) \, dt ,$$

$$= -3 \sum_{k=1}^{n} a_k \frac{\sin(kx)}{k^2} A_k \quad \text{with} \quad A_k(\epsilon) = \frac{1}{\epsilon^3} [\sin(k\epsilon) - k \epsilon \cos(k\epsilon)] ; \qquad (8)$$

Similarly:

$$\frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t \ h_n(x + t) \, dt \ \overset{(4)}{=} \ \frac{3}{2\epsilon^3} \sum_{k=1}^{n} b_k \int_{-\epsilon}^{\epsilon} t \ \sin(kx + kt) \, dt ,$$

$$= 3 \sum_{k=1}^{n} b_k \frac{\cos(kx)}{k^2} A_k . \qquad (9)$$

Therefore, the Lanczos derivative applied to partial sum (4) gives, taking $\epsilon = \frac{\pi}{n}$:

$$f'(x) = \lim_{n \to \infty} \frac{3}{2 \epsilon^3} \int_{-\epsilon}^{\epsilon} t \ f_n(x + t) \, dt ,$$

$$\overset{(8) \text{ and } (9)}{=} \lim_{n \to \infty} 3 \sum_{k=1}^{n} \frac{1}{k^2} A_k [-a_k \sin(kx) + b_k \cos(kx)] ,$$

$$= \lim{}_{n \to \infty} \sum_{k=1}^{n} \frac{3 A_k}{k^3} \frac{d}{dx} \left[a_k \cos(kx) + b_k \sin(kx) \right], \qquad (10)$$

but the Bernoulli–H$\hat{o}$pital rule permits to observe the behavior:

$$A_k \left(\epsilon = \frac{\pi}{n} \right) \xrightarrow[n >> 1]{} \frac{k^3}{3} \frac{\sin(k \epsilon)}{k \epsilon} = \frac{k^3}{3} \frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}} \overset{(6)}{=}$$

$$\frac{k^3}{3} \sigma_k , \qquad\qquad\qquad\qquad\qquad (11)$$

and this value employed in (10) implies (5), q.e.d. Thus, it is proved that the symmetric and Cioranescu-(Haslam-Jones)-Lanczos derivatives imply the same expression for the derivative of an infinite Fourier series, with the important participation of the Fejér–Lanczos factors.

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CONCLUSION

The orthogonal derivative is a generalization of the symmetric derivative and this is a generalization of the standard derivative. The orthogonal derivative acts as a smoothing (integrating filter): That is why the highest frequencies of the input are being suppressed in the deduction, and so this creates also the suppression of the Gibbs phenomenon (Lanczos 1956, 1961, 1966, Jerri 2000, Hewitt & Hewitt 1979, Gottlieb & Chi-Wang 1997, Jerri 2007). The Legendre polynomials (Lanczos 1956, 1966, Legendre 1785, Lanczos 1973) can be employed to extend the method of Cioranescu- (Haslam-Jones)-Lanczos to cover orthogonal derivatives of higher orders (Diekema & Koornwinder 2012, Rangarajan & Purushothaman 2005).

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