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Terminating $_3F_2(2)$ and Some Identities for $_2F_2$

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ABSTRACT

The Petkovsek-Wilf-Zeilberger's algorithm permits to give a simple proof of the Rathie-Pogany's identity for the hypergeometric function $_2F_2$. We exhibit elementary proofs of the formulae for $_3F_2(-N, a, 1+\lambda; 1+2a, \lambda; 2)$ and $_3F_2(-2n-1, \alpha, \beta+2; 2\alpha+2, \beta; 2)$ obtained by Kim-Choi-Rathie and Rakha-Awad-Rathie, respectively.

Keywords: Hyper-geometric function, Petkovsek-Wilf-Zeilberger's method, Kim-Choi-Rathie's equations

INTRODUCTION

Kummer (1836) obtained the property:

$$_1F_1(a, 2a; x) = e^{\frac{x}{2}}\ {}_0F_1\left(a+\frac{1}{2}; \frac{x^2}{16}\right), \tag{1}$$

that is,

$$_1F_1(a+1, 2a+2; x) = e^{\frac{x}{2}}\ {}_0F_1\left(a+\frac{3}{2}; \frac{x^2}{16}\right), \tag{2}$$

then it is natural to ask if some combination of (1) and (2) leads to a hyper-geometric function:

$$G \equiv {}_1F_1(a, 2a; x) + B\, x\ {}_1F_1(a+1, 2a+2; x). \tag{3}$$

In the second Section, we show that:

$$G = {}_2F_2(a, 1+d; 1+2a, d; x)$$

$$\text{for} \quad B = \frac{2a-d}{2(1+2a)d}, \tag{4}$$

hence equation (1) to (4) imply the Rathie-Pogany's identity (Rathie & Pogany 2008):

$$e^{-\frac{x}{2}}\ {}_2F_2(a, 1+d; 1+2a, d; x) = {}_0F_1\left(a+\frac{1}{2}; \frac{x^2}{16}\right) + \frac{2a-d}{2(1+2a)d} x\ {}_0F_1\left(a+\frac{3}{2}; \frac{x^2}{16}\right). \tag{5}$$

The relation (1) implies the values (Rainville 1960, Rakha *et al.* 2014):

$$_2F_1(-M, \beta; 2\beta; 2) = \begin{cases} \frac{(\frac{1}{2})_n}{(\beta+\frac{1}{2})_n}, & M = 2n, \quad (6) \\ 0 & M = 2n-1, \quad (7) \end{cases}$$

in terms of the Pochhammer (1890), Barnes (1908a, 1908b) symbol (shifted factorial (Koepf 1998)).

Kim *et al.* (2012) obtained the formulae:

$$_3F_2(-N, a, 1+\lambda; 1+2a, \lambda; 2) = \begin{cases} \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, & N = 2n, \quad (8) \\ \frac{\left(1-\frac{2a}{\lambda}\right)(\frac{3}{2})_n}{(1+2a)(a+\frac{3}{2})_n}, & N = 2n+1, \quad (9) \end{cases}$$

In the third Section, we employ (6) and (7) to give elementary proofs of (8) and (9).

In Rakha (2013), we observe the result:

$$A \equiv F_2(-2n-1, \alpha, \beta+2; 2\alpha+2, \beta; 2);$$

$$= \frac{(\beta-2\alpha)(\frac{3}{2})_n}{\beta(1+\alpha)(\alpha+\frac{3}{2})_n} \tag{10}$$

In the fourth Section, we use several formulae from Wolfram and the values (Rainville 1960, Rakha *et al.* 2014):

$$_2F_1(-2n-1, \lambda; 2\lambda; 2) = 0,$$

$$_2F_1(-2n, \lambda; 2\lambda; 2) = \frac{(\frac{1}{2})_n}{(\lambda+\frac{1}{2})_n}, \tag{11}$$

from (6) and (7), to give an elementary deduction of (10).

PROOF OF RATHIE-POGANY'S FORMULA

We have the expression (Rainville 1960, Buchholz 1969, Seaborn 1991):

$$_1F_1(a; c; x) = {}_1F_1(a-1; c; x) + \frac{x}{c}\ {}_1F_1(a; c+1; x), \tag{12}$$

therefore,

$$x\ {}_1F_1(a+1; 2a+2; x) = (1+2a)\left[{}_1F_1(a+1; 2a+1; x) - {}_1F_1(a; 2a+1; x)\right]. \tag{13}$$

Besides, the relation (13):

$$(a-c+1)\ {}_1F_1(a,c;x) = a\ {}_1F_1(a+1;c;x) - (c-1)\ {}_1F_1(a;c-1;x), \tag{14}$$

implies:

$$ {}_1F_1(a;2a+1;x) = 2\ {}_1F_1(a;2a;x) - {}_1F_1(a+1;2a+1;x). \tag{15}$$

If we employ (13) and (15) into (3), we get

$$G = [1-2B(1+2a)]\ {}_1F_1(a;2a;x) + 2B(1+2a)\ {}_1F_1(a+1;2a+1;x) = \sum_{k=0}^{\infty} t_k, \tag{16}$$

such that:

$$t_k = \frac{1}{k!}\left\{[1-2B(1+2a)]\frac{(a)_k}{(2a)_k} + 2B(1+2a)\frac{(a+1)_k}{(2a+1)_k}\right\}x^k, \tag{17}$$

where we use the definition of $_1F_1$ (Rainville 19603, Buchholz 1969, Seaborn 1991, Pearson 2009, Hannah 2013) in terms of the Pochhammer-Barnes symbol, whose properties permit to write:

$$(a+1)_k = \frac{a+k}{a}(a)_k\ ,\ \ (2a+1)_k = \frac{2a+k}{2a}(2a)_k\ ,$$

hence:

$$t_k = \frac{(a)_k}{k!(2a+k)(2a)_k}\{2a + [1+2B(1+2a)]\}x^k, \tag{18}$$

then (16) shall be a hyper-geometric function only if $[1+2B(1+2a)] \propto 2a$, that is:

$$1+2B(1+2a) = \frac{2a}{d}\ ,\ \ \therefore\ B = \frac{2a-d}{2(1+2a)d}\ , \tag{19}$$

and (18) adopts the form:

$$t_k = \frac{2a(a)_k(k+d)}{k!(2a)_k(k+2a)d}x^k. \tag{20}$$

In according with the algorithm of Petkovsek-Wilf-Zeilberger (Koepf 1998, Hannah 2013, Petkovsek *et al.* 1996, López-Bonilla *et al.* 2014) if the quotient of two consecutive terms in the summation (16) has the structure:

$$\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\cdots(k+a_p)\,x}{(k+b_1)\cdots(k+b_q)(k+1)}, \tag{21}$$

then(16) is connected with $_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x)$; in Petkovsek *et al.* (1996), we find details of how to use Mathematica and Maple programs to convert from the *k*-th term of a series to a description of the corresponding hypergeometric function. From (20) we have $t_0 = 1$ and:

$$\frac{t_{k+1}}{t_k} = \frac{(k+a)(k+1+d)\,x}{(k+1+2a)(k+d)(k+1)}, \tag{22}$$

and the comparison with (21) implies (4) hence the Rathie-Pogany's identity (Rathie & Pogany 2008, Kim *et al.* 2012) indicated in (5). QED.

A similar process permits to show the Rakha-Awad-Rathie's formula (Rakha *et al.* 2014, Rakha *et al.* 2013):

$$e^{-\frac{x}{2}}\ {}_2F_2(a,2+d;2+2a,d;x) = {}_0F_1\left(a+\frac{3}{2};\frac{x^2}{16}\right) + \frac{2a-d}{2(1+a)d}x\ {}_0F_1\left(a+\frac{3}{2};\frac{x^2}{16}\right) + \frac{c}{2(2a+3)}x^2\ {}_0F_1\left(a+\frac{5}{2};\frac{x^2}{16}\right),\ \ c = \frac{1}{d}\left[\frac{a}{1+d} - \frac{2a-d}{2(1+a)}\right]. \tag{23}$$

KIM-CHOI-RATHIE'S RELATION FOR $_3F_2(2)$

With the definition of the hypergeometric function $_3F_2$ (Pearson 2009) is simple to deduce the interesting relation:

$$\frac{d}{d\lambda}\ {}_3F_2(-N,a,1+\lambda;1+2a,\lambda;2) = \frac{2aN}{(1+2a)\lambda^2}\ {}_2F_1(1-N,1+a;2+2a;2), \tag{24}$$

hence it is convenient to consider two cases:

Case I: $N = 2n$

From (24), we have

$$\frac{d}{d\lambda}\ {}_3F_2(-2n,a,1+\lambda;1+2a,\lambda;2) = \frac{4an}{(1+2a)\lambda^2}\ {}_2F_1(-(2n-1),1+a;2(1+a);2) = 0,$$

by (7) with $M = 2n-1$ and $\beta = 1+a$. Therefore $_3F_2(-2n,a,1+\lambda;1+2a,\lambda;2)$ is independent of the parameter λ, and thus we can calculate it via some value of λ, for example $\lambda = 2a$:

$$\begin{aligned} {}_3F_2(-2n,a,1+\lambda;1+2a,\lambda;2) &= {}_3F_2(-2n,a,1+2a;1+2a,2a;2) \\ &= {}_2F_1(-2n,a;2a;2), \\ &= \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n} = (4),\ \ QED. \end{aligned}$$

where we use the result (6) with $M = 2n$ and $\beta = a$.

Case II: $N = 2n+1$

Again from (24), we have

$$\frac{d}{d\lambda}\ {}_3F_2(-2n-1,a,1+\lambda;1+2a,\lambda;2) = \frac{2a(2n+1)}{(1+2a)\lambda^2}\ {}_2F_1(-2n,1+a;2(1+a);2),(6)$$

$$= \frac{2a(2n+1)(\frac{1}{2})_n}{(1+2a)(a+\frac{3}{2})_n\lambda^2} = \frac{2aQ}{\lambda^2}\ ,\ Q = \frac{(\frac{3}{2})_n}{(1+2a)(a+\frac{3}{2})_n}\ ,$$

whose integration gives:

$$ {}_3F_2(-2n-1,a,1+\lambda;1+2a,\lambda;2) = -\frac{2aQ}{\lambda} + A, \tag{25}$$

where A is independent of λ. Then we can determine A with $\lambda = 2a$:

$$ {}_3F_2(-2n-1,a,1+2a;1+2a,2a;2 = {}_2F_1(-2n-1,a,2a;2) = 0 = -Q + A,$$

that is, $A = Q$, hence (25) implies (9). QED.

Our process shows that the Kim-Choi-Rathie's formulae can be deduced, in elementary manner, from the relations (6), (7) and (24).

RAKHA-AWAD-RATHIE'S EXPRESSION FOR $_3F_2$ (2)

In Wolfram are the following relations for hyper-geometric functions:

$$b\ {}_3F_2(a,a_2,a_3;b,b_2;z) - a\ {}_3F_2(a+1,a_2,a_3;b+1,b_2;z) + (a-b)\ {}_3F_2(a,a_2,a_3;b+1,b_2;z) = 0, \quad (26)$$

$$c(a-b)\ {}_3F_2(a,b,a_3;\ c,\ b_2;z) - a(c-b)\ {}_3F_2\ (a+1,b,a_3;c+1,b_2;z) + \\ +b(c-a)\ {}_3F_2(a,b+1,a_3;c+1,b_2;z) = 0, \quad (27)$$

$$(a-c+1)(b-c+1)\ z\ {}_2F_1(a,b;c;z) + (c-1)(c-2)(z-1)\ {}_2F_1(a,b;c-2;z) + \\ +(c-1)[(a+b-2c+3)z+c-2]\ {}_2F_1(a,b;c-1;z) = 0, \quad (28)$$

$$ {}_2F_1(-n,\alpha;2\alpha+1;2) = \frac{\Gamma(\alpha+\frac{1}{2})}{2\sqrt{\pi}}\left[\frac{(1+(-1)^n)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\alpha+\frac{n+1}{2}\right)} + \frac{(1-(-1)^n)\Gamma(\frac{n+2}{2})}{\Gamma(\alpha+\frac{n+2}{2})}\right] \quad (29)$$

which permit to exhibit an elementary proof of (10). In fact, if into (26) we employ $a=-2n-1$, $a_2=\alpha$, $a_3=\beta+2$, $b=\beta$, $b_2=2\alpha+2$ and $z=2$, we obtain the expression:

$$\beta A = (2n+\beta+1)\ B - (2n+1)\ C, \quad (30)$$

such that

$$B \equiv\ {}_3F_2(-2n-1,\alpha,\beta+2;\beta+1{,}2\alpha+2;2),$$
$$C \equiv\ {}_3F_2(-2n,\alpha,\beta+2;\beta+1{,}2\alpha+2;2). \quad (31)$$

Now we must determine the quantities *B* and *C*, hence in (27) we introduce the values $a=-2n-1, b=\alpha$, $c=\beta+1, a_3=\beta+2, b_2=2\alpha+2$ and $z=2$ to deduce the relation:

$$B = \frac{(2n+1)(\beta+1-\alpha)}{(2n+\alpha+1)(\beta+1)}\ {}_2F_1(-2n,\alpha;2\alpha+2;2), \quad (32)$$

where was applied the first formula(11) with $\lambda=\alpha+1$.

If into (28), we use $a=-2n, b=\alpha, c=2\alpha+2$ and $z=2$ we obtain that:

$$\frac{\alpha+1}{2\alpha+1}\ {}_2F_1(-2n,\alpha;2\alpha+2;2) =\ {}_2F_1(-2n,\alpha;2\alpha+1;2) - \frac{\alpha}{2n+2\alpha+1}\ {}_2F_1(-2n,\alpha;2\alpha;2), \quad (33)$$

besides from (11) and (29) are immediate the values:

$$ {}_2F_1(-2n,\alpha;2\alpha;2) =\ {}_2F_1(-2n,\alpha;2\alpha+1;2) = \frac{(\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n}, \quad (34)$$

hence from (32) and (33):

$$ {}_2F_1(-2n,\alpha;2\alpha+2;2) = \frac{(2n+\alpha+1)\left(\frac{3}{2}\right)_n}{(2n+1)(\alpha+1)\left(\alpha+\frac{3}{2}\right)_n},$$

$$B = \frac{(\beta+1-\alpha)\,(\frac{3}{2})_n}{(\beta+1)(\alpha+1)\,(\alpha+\frac{3}{2})_n}. \quad (35)$$

To calculate *C*, into (27) we employ $a=-2n$, $b=\alpha, c=\beta+1, a_3=\beta+2, b_2=2\alpha+2$ and $z=2$, thus:

$$C = \frac{2}{(2n+\alpha)(\beta+1)}[n(\beta+1-\alpha)\ {}_2F_1(-2n+1,\alpha;2\alpha+2;2) + \frac{\alpha}{2}(2n+\beta+1)\ {}_2F_1(-2n,\alpha+1;2\alpha+2;2)] \quad (36)$$

and from (11) with $\lambda=\alpha+1$:

$$ {}_2F_1(-2n,\alpha+1;2\alpha+2;2) = \frac{(\frac{1}{2})_n}{(\alpha+\frac{3}{2})_n} \quad (37)$$

besides from (28) with $a=-2n+1, b=\alpha, c=2\alpha+2$ and $z=2$:

$$ {}_2F_1(-2n+1,\alpha;2\alpha+2;2) = \frac{2\alpha+1}{\alpha+1}\ {}_2F_1(-2n+1,\alpha;2\alpha+1;2) = \frac{(2n+2\alpha+1)\,(\frac{3}{2})_n}{(2n+1)(\alpha+1)\,(\alpha+\frac{3}{2})_n}. \quad (38)$$

Now we put (37) and (38) into (36) to obtain:

$$C = \frac{(2n+2\alpha+1)\left(\frac{3}{2}\right)_n}{(2n+1)(2n+\alpha)(\beta+1)\left(\alpha+\frac{3}{2}\right)_n}\left[\frac{\alpha(2n+\beta+1)}{2n+2\alpha+1} + \frac{2n(\beta+1-\alpha)}{\alpha+1}\right], \quad (39)$$

Finally, the application of *(35) and (39) in (30)* permits to construct the formula (10) deduced by Rakha *et al.* (2013). QED

CONCLUSION

The calculations here exhibited show that the formulae of Kummer (1836), Wolfram system and the Petkovsek-Wilf -Zeilberger's algorithm (Koepf 1998, Hannah 2013, López-Bonilla *et al.* 2014, Petkovsek *et al.* 1996) allow to deduce the identities published by Rathie and Pogany (2008), Rakha *et al.* (2014), Rakha *et al.*(2013) and Kim *et al.* (2012).

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