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Theory of Finsler spaces with $(\lambda,\beta)-$Metric

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ABSTRACT

The aim of this paper is to introduce and study the concept of $(\lambda,\beta)-$metric and a number of propositions and theorems have been workout for a $(\lambda,\beta)-$metric, where $\lambda^m = a_{i_1,i_2,\cdots,i_m} y^{i_1} y^{i_2} \cdots y^{i_m}$ is a m^{th} -root metric and $\beta = b_i(x)y^i$ is a one form metric.

Keywords: Finsler space with $(\lambda,\beta)-$metric, m^{th} -root metric, one form metric, curvature tensor, torsion tensor.

INTRODUCTION

We begin with recalling some notations and basic definitions that are used in this paper.

Definition 1: Finsler space

Suppose that we are given a function $L(x^i, y^i)$ of the line element (x^i, y^i) of a curve defined in R. We shall assume L as a function of class at least C^5 in all its 2n-arguments. If we define the infinitesimal distance ds between two points $P(x^i)$ and $Q(x^i + dx^i)$ of R by the relation $ds = L(x^i, dx^i)$ then, the manifold M^n equipped with the fundamental function L defining the metric is called a Finsler space if $L(x^i + dx^i)$ satisfies the following condition:-

A -The function $L(x^i, y^i)$ is positively homogeneous of degree one in y^i i. e.,

$L(x^i, ky^i) = k\,L(x^i, y^i), \qquad k > 0$

That is, the arc length of curve is independent of the choice of parameter t.

B -The function $L(x^i, y^i)$ is positive if not all y^i vanish simultaneously, i.e.,

$$L(x^i, y^i) > 0 \quad \text{with} \quad \sum_i (y^i)^2 \neq 0$$

That is, the distance between two distinct points is positive.

C- The quartic form,

$$\dot\partial_i \dot\partial_j L^2(x^i, y^i)\xi^i\xi^j = \frac{\partial^2 L^2(x^i, y^i)}{\partial y^i \partial y^j}\xi^i\xi^j$$ is assumed to be positive definite for all any variable ξ^i.

That is, $L(x^i, y^i)$ is a convex function in y^i (Matsumoto, 1986)

Definition 2: Finsler connection

A Finsler connection $F\Gamma$ is defined as tried $(F^i_{jk}(x,y), N^i_j(x,y), V^i_{jk}(x,y))$ as follows:

(i) $F^i_{jk}(x,y)$ are called the connection coefficients of h-connection which obey the usual transformation law of connection coefficients of a connection

$$\bar F^a_{bc}(\bar x,\bar y) = F^i_{jk}(x,y)\bar X^a_i \underline{X}^j_b \underline{X}^k_c + \bar X^a_i \partial_c \underline{X}^i_b$$

('h' is the abbreviation of 'horizontal'.)

(ii) $N^i_j(x,y)$ are called the connection coefficients of non-linear connection which obey the transformation law

$$\bar N^a_b(\bar x,\bar y) = N^i_j(x,y)\bar X^a_i \underline{X}^j_b + \bar X^a_i \partial_b \underline{X}^i_c \bar y^c$$

(iii) $V^i_{jk}(x,y)$ are called the connection coefficients of v-connection which are components of a tensor field of (1,2)-ype. ('v' is the abbreviation of 'vertical'.) (Antonelli *et al.* 1993).

Definition 3: h and v- covariant derivative,

A tensor field $T^i_j(x,y)$, for instance, of (1,1)-type we have first the h-covariant derivative $''\,^h T$ whose components are given by

$$T^i_{j|k} = \delta_k T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk} \quad \ldots\ldots\ldots\ldots(1)$$

where, δ_k is a differential operator $\delta_k = \partial_k - N^r_k \dot\partial_r$.

Secondly, we have the v-covariant derivative $''\,^v T$ whose components are given by

$$T^i_j|_k = \dot\partial_k T^i_j + T^r_j V^i_{rk} - T^i_r V^r_{jk} \quad \ldots\ldots\ldots\ldots(2)$$

(Matsumoto, 1986)

Definition 4: Berwald connection

The Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is uniquely determined from function L(x, y) of F^n by the following five axioms:-

$L_{|i} = 0$, i.e. L -metrical …………………(B1)

(h) h- torsion: $T^i_{jk} = 0$ ………………………….(B2)

deflection: $D^i_j = 0$ ……………………………(B3)

(v) hv -torsion: $P^i_{jk} = 0$ ……………………………(B4)

(h) hv -torsion: $C^i_{jk} = 0$ ……………………………(B5)

(Matsumoto, 1992).

Definition 5: The Cartan connection $C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ is uniquely determined from function L(x, y) of F^n by the following five axioms:-

$g_{ij|k} = 0$ i.e. h- metrical ……………….(C1)

(h) h -torsion: $T^i_{jk} = 0$ ………………….(C2)

deflection tensor field $D^i_j = 0$ ………………….(C3)

$g_{ij}|_k = 0$, i.e. v -metrical ……………………….(C4)

(h) h -torsion: $S^i_{jk} = 0$. ………………………(C5)

(Matsumoto, M. 1992).

Matsumoto (1995) introduce the concept of m^{th}-root metric on a differentiable manifold with the local co-ordinate x^i, is defined by

$$L(x,y) = \{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} y^{i_3} \cdots y^{i_m}\}^{1/3} \quad \ldots\ldots\ldots(3)$$

where, $a_{i_1 i_2 i_3 \cdots i_m}(x)$ are components of a symmetric tensor field of (0, m)-type depending on the position x alone, and a Finsler space with a m^{th}-root metric is called the *m^{th}-root Finsler space* (Shimada, 1979; Kropina 1961; Wagner, 1938; Wagner, 1943). They studying the m^{th}-root Finsler spaces with a cubic metric as a generalization of Euclidean or Riemannian geometry. In 1941, Randers co-relate the Finsler Theory with the unified field theory of gravitation and electromagnetism. The formula for the length ds of a line-element dx^i must necessarily be homogeneous of first degree in dx^i. The simplest "eccentric" line-element possessing this property, and of course being invariant, is

$$ds = \{a_{ij}(x) dx^i dx^j\}^{1/2} + b_i(x) dx^i \quad \ldots\ldots\ldots\ldots(4)$$

Where $a_{ij}(x)$ is the fundamental tensor of the Riemannian affine connection, and b_i is a covariant vector determining the displacement of the centre of the indicatrix."

In the present paper, we shall study Finsler space with the fundamental function $L(\lambda,\beta)$. We have workout some basic tensors namely h_{ij}, g_{ij}, C_{ijk} and g^{ij} and also workout certain propositions regarding the Finsler space with (λ,β)-metric.

Basic tensors of (λ,β)-metric

Definition 6: A Finsler metric L(x, y) is called a (λ,β)-metric, when L is positively homogeneous function $L(\lambda,\beta)$ of first degree in two variables, λ and β, where $\lambda^m = a_{i_1 i_2 i_3 \ldots i_m}(x) y^{i_1} y^{i_2} y^{i_3} \ldots y^{i_m}$ is m^{th} root metric and $\beta = b_i(x) y^i$ is a one-form metric (Matsumoto &Shimada, 1978).

Throughout, the present paper, following notations are adopted

$$a_k = a_{i_1 i_2 \ldots i_{m-1}}(x) y^{i_1} y^{i_2} y^{i_3} \cdots y^{i_{m-1}}$$

$$a_{ij} = (m-1) a_{ij i_3 i_4 \ldots i_{m-2}}(x) y^{i_3} y^{i_4} \cdots y^{i_{m-2}}$$

$$a^{ij} b_j = B^i \text{ and } a^{ij} a_j = a^i$$

where, (a^{ij}) is the inverse matrix of (a_{ij}). Further, subscripts λ,β denote partial differentiations with respect to λ,β respectively. As for a (λ,β)-metric, $L = L(\lambda,\beta)$ ……………………………(5)

Differentiating (5), with respect to y^i, we get

$$l_i = \dot{\partial}_i L = \frac{L_\lambda}{\lambda^{m-1}} a_i + L_\beta b_i \quad \ldots\ldots\ldots\ldots(6)$$

Equation (6) can also be written as

$$y_i = L l_i = L(\dot{\partial}_i L) = \frac{L L_\lambda}{\lambda^{m-1}} a_i + L L_\beta b_i \quad \ldots\ldots\ldots(7)$$

Again differentiating (6), with respect to, y^j, we have the angular metric tensor $h_{ij} = L(\dot{\partial}_i \dot{\partial}_j L)$, as

$$h_{ij} = p_{-(m-2)} a_{ij} + q_0 b_i b_j + q_{-(m-1)}(a_i b_j + a_j b_i) + q_{-(2m-2)} a_i a_j \quad \ldots.. (8)$$

where, $p_{-(m-2)} = \frac{L L_\lambda}{\lambda^{m-1}}$, $q_0 = L L_{\beta\beta}$, $q_{-(m-1)} = \frac{L L_{\lambda\beta}}{\lambda^{m-1}}$,

$$q_{-(2m-2)}=\frac{L}{\lambda^{2m-2}}\left(L_{\lambda\lambda}-\frac{(m-1)L_\lambda}{\lambda}\right) \quad \text{..................(8)'}$$

Owing to the homogeneity or $h_{ij}y^j=0$, we have two identities,

$$\begin{cases} p_{-(m-2)}+q_{-(m-1)}\beta+q_{-(2m-2)}\lambda^m=0 \\ q_0\beta+q_{-(m-1)}\lambda^m=0 \end{cases} \quad \text{..................(9)}$$

Since a_i and b_i are two independent vector fields, hence, we must have.

Again,

$$g_{ij}=h_{ij}+l_il_j \quad \text{..................(10)}$$

$$g_{ij}=p_{-(m-2)}a_{ij}+p_0b_ib_j+p_{-(m-1)}(a_ib_j+a_jb_i)+p_{-(2m-2)}a_ia_j$$

where, $p_0=q_0+L_\beta^2$, $p_{-(m-1)}=q_{-(m-1)}+\dfrac{L_\lambda L_\beta}{\lambda^{m-1}}$,

$$p_{-(2m-2)}=q_{-(2m-2)}+\frac{L_\lambda^2}{\lambda^{2m-2}} \quad \text{..................(10)'}$$

Using (9) and (10)', we get

$$\begin{cases} p_0\beta+p_{-(m-1)}\lambda^m=LL_\beta \\ p_{-(m-1)}\beta+p_{-(2m-2)}\lambda^m=0 \end{cases} \quad \text{..................(11)}$$

It is well known that

Proposition 1: Let a non-singular symmetric n-matrix (A_{ij}) and n quantities c_i be given, and put $B_{ij}=A_{ij}+c_ic_j$. The inverse matrix (B^{ij}) of (B_{ij}) and the det(B_{ij}) are given by,

$$B^{ij}=A^{ij}-\frac{1}{(1+c^2)}c^ic^j, \quad \det(B_{ij})=A(1+c^2)$$

where, (A^{ij}) is the inverse matrix of (A_{ij}), $A=\det(A_{ij})$, $c^i=A^{ij}c_j$, and $c^2=c^ic_i$ (Matsumoto, 1992).

From (10), the components g_{ij} may be written as,

$$g_{ij}=p_{-(m-1)}a_{ij}+c_ic_j+d_id_j$$

where, we put,

$$c_i=\pi b_i,\ d_i=\pi_0 b_i+\pi_{-(m-1)}a_i$$

$$\pi^2+\pi_0^2=p_0\,.\,\pi_0\pi_{-(m-1)}=p_{-(m-1)},$$

$$\pi_{-(m-1)}^2=p_{-(2m-2)}.$$

we get $B_{ij}=p_{-(m-2)}a_{ij}+c_ic_j$, then we have, $g_{ij}=B_{ij}+d_id_j$.

From, definition of B^{ij}, we have $B_{ij}B^{jk}=\delta_i^k$.

Then,

$$B^{ij}=\frac{1}{p_{-(m-2)}}\left(a^{ij}-\frac{c^ic^j}{p_{-(m-2)}+c^2}\right)$$

where, a^{ij} is reciprocal of a_{ij}, $c^i=a^{ij}c_j$, and $c^2=c^ic_i$. Now, by using Proposition 1, we have

$$g^{ij}=B^{ij}-\frac{d^id^j}{1+d^2}$$

where, $d^i=B^{ij}d_j$, $d^id_i=d^2$

$$|g_{ij}|=|B_{ij}|(1+d^2)=|p_{-(m-2)}a_{ij}|\frac{(p_{-(m-2)}+c^2)}{p_{-(m-2)}}(1+d^2)$$

$$|g_{ij}|=p_{-(m-2)}^{n-1}a(p_{-(m-2)}+c^2)(1+d^2)$$

where a is the determinant of a_{ij}.

$$g^{ij}=\frac{1}{p_{-(m-2)}}a^{ij}-\frac{c^ic^j}{p_{-(m-2)}(p_{-(m-2)}+c^2)}-\frac{d^id^j}{1+d^2}.$$

Now,

$$d^i=B^{ij}d_j=\frac{1}{p_{-(m-2)}}\left[\frac{(\pi_0p_{-(m-2)}-\pi^2\pi_{-(m-1)}\bar{a})}{(p_{-(m-2)}+c^2)}B^i+\pi_{-(m-1)}a^i\right]$$

where, $B^ib_i=b^2=a^{im}b_mb_i$,

$a_iB^i=a^{im}a_ib_m=a^ib_i=\bar{a}$, $\quad \pi^2b^2=c^2$.

Again, $$d^id^j=-\frac{1}{p_{-(m-2)}^2}\left[\frac{(\pi_0p_{-(m-2)}-\pi^2\pi_{-(m-1)}\bar{a})^2}{(p_{-(m-2)}+c^2)^2}B^iB^j\right.$$

$$\left.+\frac{(\pi_0p_{-(m-2)}-\pi_0^2\pi_{-(m-1)}\bar{a})}{(p_{-(m-2)}+c^2)}\pi_{-(m-1)}(B^ia^j+B^ja^i)+\pi_{-(m-1)}^2a^ia^j\right]$$

Or $$d^id^j=\frac{1}{p_{-(m-2)}^2}\left[\frac{(\pi_0p_{-(m-2)}-\pi^2\pi_{-(m-1)}\bar{a})^2}{(p_{-(m-2)}+c^2)^2}B^iB^j\right.$$

$$\left.+\frac{(p_{-(m-1)}p_{-(m-2)}-\pi^2p_{-(2m-2)}\bar{a})}{(p_{-(m-2)}+c^2)}(B^ia^j+B^ia^i)+p_{-(2m-2)}a^ia^j\right].$$

Now,

$$d^2 = d_i d^i = \frac{1}{p_{-(m-2)}(p_{-(m-2)} + c^2)}[\pi_0^2 b^2 p_{-(m-2)} +$$

$$2p_{-(m-1)}p_{-(m-2)}\bar{a} - p^2_{-(m-1)}\bar{a}^2$$

$$+p_{-(m-2)}p_{-(2m-2)}a^2 + p_{-(2m-2)}c^2a^2].$$

Again,

$$|g_{ij}| = p^{n-1}_{-(m-2)}a(p_{-(m-2)} + c^2)(1+d^2) = p^{n-1}_{-(m-2)}a\tau$$

where,

$$\tau = p_{-(m-2)}\{p_{-(m-2)} + \pi_0^2 b^2 + p_{-(m-2)}\bar{a}\} +$$

$$\{p_{-(m-1)}p_{-(m-2)}\bar{a} - p^2_{-(m-1)}\bar{a}^2\}$$

$$+\{p_{-(m-2)}p_{-(2m-2)}a^2 + p_{-(2m-2)}a^2c^2\}.$$

Thus, the reciprocal of (g_{ij}) is given by

$$g^{ij} = \frac{1}{p_{-(m-2)}}a^{ij} - SB^iB^j - S(B^ia^j + B^ja^i) - Sa^ia^j \quad \text{…………(12)}$$

where,

$$S_{2m-4} = \frac{\pi_0 p^2_{-(m-2)} + \pi^2(\tau + \pi^2 p_{-(2m-2)}\bar{a}^2 - 2p_{-(m-1)}p_{-(m-2)}\bar{a})}{\tau p_{-(m-2)}\{p_{-(m-2)} + c^2\}}$$

$$S_{m-3} = \frac{p_{-(m-1)}p_{-(m-2)} - \pi^2 p_{-(2m-2)}\bar{a}}{\tau p_{-(m-2)}}$$

$$S_{-(2m-4)} = \frac{p_{-(m-2)}p_{-(2m-2)} + c^2 p_{-(2m-2)}}{\tau p_{-(m-2)}} \quad \text{………..(12)'}$$

Differentiating (12) by y^k, we get,

$$2C_{ijk} = 2p_{-(m-2)}a_{ijk} + p_{0\beta}b_ib_jb_k + \prod_{(ijk)}\{P_ia_{jk} + p_{-(m-1)\beta}a_ib_jb_k$$

$$+\frac{p_{-(m-1)\lambda}}{\lambda^{m-1}}a_ia_jb_k\} + \frac{p_{-(2m-2)}}{\lambda^{m-1}}a_ia_ja_k \quad \text{……..…..(13)}$$

where, $\prod_{(ijk)}$ denotes the sum of cyclic permutation of i, j, k.

$$P_i = p_{-(2m-2)}a_i + p_{-(m-1)}b_i$$

$$2p_{-(m-2)}c_{ijk} = 2p^2_{-(m-2)}a_{ijk} + r_{-(m-1)}b_ib_jb_k$$

or,

$$+\prod_{(ijk)}\{p_ih_{jk} + r_{-(2m-2)}a_ib_jb_k$$

$$+ r_{-(3m-3)}a_ia_jb_k\} + r_{-(4m-4)}a_ia_ja_k \quad \text{……. (13)}$$

where,

$$r_{-(m-1)} = p_{0\beta}p_{-(m-2)} - 3q_0p_{-(m-1)}, \quad \text{…………(13a)'}$$

$$r_{-(2m-2)} = p_{-(m-1)\beta}p_{-(m-2)} - q_0p_{-(2m-2)} - 2p_{-(m-1)}q_{-(m-1)}$$

$$r_{-(3m-3)} = \frac{p_{-(m-1)\lambda}p_{-(m-2)}}{\lambda^{m-1}} - q_{-(2m-2)}p_{-(m-1)} - 2p_{-(2m-2)}q_{-(m-1)},$$

$$r_{-(4m-4)} = \frac{p_{-(m-2)}p_{-(2m-2)\lambda}}{\lambda^{m-1}} - 3q_{-(2m-2)}p_{-(2m-2)}$$

Proposition 2: The normalized supporting element l_i, angular metric tensor h_{ij}, metric tensor g_{ij} and (h) hv-torsion tensor C_{ijk} of Finsler space with (λ, β)-metric are given by (6), (8), (10) and (13a) respectively.

Proposition 3: The reciprocal of the metric tensor g_{ij} of (λ, β)-metric is given by (12).

Proposition 4: The coefficients $r_{-(m-1)}, r_{-(2m-2)}, r_{-(3m-3)}, r_{-(4m-4)}$ defined in (13a)' satisfy the following relation

$$r_{-n(m-1)}\beta + r_{-(n+1)(m-1)}\lambda^m = 0, \quad \text{…….. ………...(14)}$$

where $n = 1, 2, 3$.

Now, from (11) and (14), we have

$$p_{-(2m-2)} = \phi p_{-(m-1)},$$

$$r_{-(n+1)(m-1)} = \phi^n r_{-(m-1)}, n = 1, 2, 3 \quad \text{…………….15)}$$

where, $\phi = -\frac{\beta}{\lambda^m}$.

Using relation (15) in (13a), we easily get

$$2p_{-(m-2)}C_{ijk} = 2p^2_{-(m-2)}a_{ijk} + \prod_{(ijk)}(H_{jk}P_i) \quad \text{……. ... (13b)}$$

where, $H_{ij} = h_{ij} + \frac{r_{-(m-1)}}{3p^3_{-(m-1)}}P_iP_j$.

Further, by direct computation from C_{ijk} and g^{ij}, we have,

$$C_i = p_{-(m-2)}a_{ijk}g^{jk} + Aa_i + Bb_i \quad \text{…………….……(16)}$$

where A and B are certain scalar.

If the Finsler space (F^n) is C-reducible, then

$$C_{ijk} = \frac{1}{(n+1)}\prod_{(ijk)}(h_{ij}C_k) \quad \text{………………………..(17)}$$

from (13b) and (17), it follows that,

$$a_{ijk} + \frac{r_{-(m-1)}}{2p^3_{-(m-1)}p^2_{-(m-2)}}P_iP_jP_k = \prod_{(ijk)}h_{ij}N_k \quad \text{……...…..(18)}$$

where, $N_k = \frac{2p_{-(m-2)}}{n+1}C_k - P_k$. Conversely, If (18), is satisfied for certain covariant vector N_k, then from 3b) we have

$$2p_{-(m-2)}C_{ijk} = \prod_{(ijk)}(h_{ij}(P_k + N_k)) \quad \text{.......(19)}$$

which gives (17). Thus, we have,

Theorem 1: A Finsler space with (λ,β)-metric is C-reducile iff equation (18) holds.

Important tensors of (λ,β)-metric

It follow from (13b) and (12), that the components C^i_{jk} of the (h)hv-torsion tensor $C\Gamma$ are given by,

$$2p_{-(m-2)}C^i_{jk} = 2p^2_{-(m-2)}a^i_{jk} + (\delta^i_j P_k - l^i l_j P_k) + (\delta^i_k P_j - l^i l_k P_j) + \frac{r_{-(m-1)}}{p^3_{-(m-1)}}P^i P_j P_k + h_{jk}P^i \quad \text{..........(20)}$$

where, $P_i g^{ik} = P^j$, $g^{ij}l_j = l^i$, $a_{rjk}g^{ri} = a^i_{jk}$.

Again from (13b) and (20), we have,

$$4p^2_{-(m-2)}C^r_{hk}C_{rij} = 4p^4_{-(m-2)}a^r_{hk}a_{rij} + 2p^2_{-(m-2)}\prod_{(ijk)}(a_{ijk}P_h) - \frac{r_{-(m-1)}\bar{P}}{p^3_{-(m-1)}L^2}\prod_{(ijkh)}(P_i P_j P_k l_h) - \frac{2p^2_{-(m-2)}}{L}[a_{ij}(l_h P_k + l_k P_h)]$$

$$\frac{2p^2_{-(m-2)}r_{-(m-1)}}{p^3_{-(m-1)}} a_{rij}p^r p_h p_k + \frac{2p^2_{-(m-2)}r_{-(m-1)}}{p^3_{-(m-1)}} a^r_{hk}p_r p_i p_j +$$

$$p^2_{-(m-2)}a_{rij}p^r h_{hk} + 2\, p^2_{-(m-2)}a^r_{hk}p_r h_{ij} -$$

$$p^2_{-(m-2)}[a^r_{hk}\, l_r(l_i P_j + l_j P_i)] + [h_{ih}p_j p_k + h_{ik}p_h p_j + h_{hj}P_k P_i + h_{jk}P_i P_h + 2h_{ij}P_k P_h + 2h_{hk}P_i P_j)]$$

$$+\frac{r_{-(m-1)}}{p^3_{-(m-1)}}\frac{P^2}{L^4}[h_{ij} + \frac{r_{-(m-1)}}{p^3_{-(m-1)}}P_i P_j]P_k P_h - \frac{\bar{P}}{L^2}h_{ij}(l_h P_k + l_k P_h)$$

$$-\frac{\bar{P}}{L^2}h_{hk}(l_i P_j + l_j P_i) + \frac{r_{-(m-1)}}{p^3_{-(m-1)}}\frac{P^2}{L^4}h_{hk}P_i P_j + \frac{4r_{-(m-1)}}{p^3_{-(m-1)}}P_i P_j P_k P_h. \quad \text{...........(21)}$$

where, $a_{irk}l^r = \frac{a_{ik}}{L}$, $P^r l_r = \frac{\bar{P}}{L^2}$, $P^r P_r = \frac{P^2}{L^4}$, $P^r g_{ir} = P_i$, $\delta^r_i a_{rhk} = a_{ihk}$.

From (21), the v-curvature tensor S_{hijk} of $C\Gamma$ is written as, $4p^2_{-(m-2)}S_{hijk} = 4p^2_{-(m-2)}\underset{(jk)}{\Theta}(C^r_{hk}C_{rij})$

where, $\underset{(jk)}{\Theta}$ anti-symmetric with respect to indecies j and k. Thus,

$$4p^2_{-(m-2)}S_{hijk} = \underset{(jk)}{\Theta}[4p^4_{-(m-2)}a^r_{hk}a_{rij} + 2p^2_{-(m-2)}(a_{rij}P_r H_{hk} + a^r_{hk}P_r H_{ij}) - (l_h P_k + l_k P_h)A_{ij} - (l_i P_j + l_j P_i)A_{hk} + H'_{ij}P_h P_k + H'_{hk}P_i P_j] \quad \text{....... (22)}$$

where,

$$A_{ij} = 2p^2_{-(m-2)}a_{ij} - \frac{\bar{P}}{L^2}h_{ij}$$

$$H'_{ij} = 2p^2_{-(m-2)}\frac{2r_{-(m-1)}}{3p^3_{-(m-1)}}a_{rij}P_r + \left(1 + \frac{P^2}{L^4}\right)h_{ij}.$$

Proposition 5: The v-curvature tensor of a Finsler space with (λ,β)-metric is given by (22).

Next, h- and v-covariant derivatives $X_{i|j}, X_i|_j$ of a covariant vector field X_i with respect to the Cartan connection $C\Gamma$ are defined by,

$X_{i|j} = \partial_j X_i - (\dot{\partial}_r X_i)N^r_j - X_r F^r_{ij}$ and $X_i|_j = \dot{\partial}_j X_i - X_r C^r_{ij}$

where, $(F^i_{jk}, N^i_j(=F^i_{0j}), C^i_{jk})$ are connection coefficients of $C\Gamma$ and suffix '0' means the contraction by supporting element y^i (Antonelli *et al.* 1993).

If $b_{i|h} = 0$, then for $L(\lambda,\beta)$-metric, we have,

$$a_{i|j} = 0 \quad , \quad a_{ij|k} = 0 \quad \text{......................(23)}$$

because, $l_{i|j} = 0$ and $h_{ij|k} = 0$.

Then, the h-covariant differentiation of (13b), we have,

$$C_{ijk|h} = p_{-(m-2)}a_{ijk|h} \quad \text{.....................(24)}$$

Therefore, the v(hv)-torsion tensor P_{ijk} is written as,

$$P_{ijk} = C_{ijk|h}y^h = C_{jk|0} = p_{-(m-2)}a_{ijk|0} \quad \text{.............(25)}$$

Definition: 7 A Finsler space is called a Berwald space, if $C_{ijk|h}$ vanishes identically and called a Landsberg space if $C_{ijk|0}$ vanishes identically.

Theorem 2: If b_i is h-covariantly constant (resp. $b_{i|0}$), then a Finsler space with (λ,β)-metric is a Berwald space (resp. Landsberg space) iff the tensor $a_{ijk|h}$ (resp. $a_{ijk|0}$) vanishes identically.

Now, the hv-curvature tensor P_{hijk} (Antonelli *et al.* 1993] is given by,

$$P_{hijk} = \underset{(hi)}{\Theta}(C_{ijk|h} + C^r_{hj}C_{rik|0}).$$

Now,

$$C^r_{hj}C_{rik|0} = p^2_{-(m-2)}a^r_{hj}a_{rik|0} + \frac{1}{2}a_{hik|0}P_j -$$
$$\frac{1}{2L}a_{ik|0}(l_hP_j + l_jP_h) + \frac{1}{2}a_{jik|0}P_h$$
$$+\frac{r_{-(m-1)}}{2p^3_{-(m-1)}}a_{rik|0}P^rP_jP_h + \frac{1}{2}h_{jh}P^ra_{rik|0}.$$

Thus,

$$P_{hijk} = \underset{(hi)}{\Theta}[a_{ijk|h} + \frac{1}{2}a_{ijk|0}P_h - \frac{1}{2L}a_{ik|0}(l_hP_j + l_jP_h) \quad \text{.........(26)}$$
$$+a_{rik|0}P^rH_{jh} + a_{rik|0}A^r_{hj}]$$

where, $A^r_{hj} = p^2_{-(m-2)}a^r_{hj} + \frac{1}{b}\frac{r_{-(m-1)}}{p^3_{-(m-1)}}P^rP_jP_h$.

Proposition 6: The (v) hv-torsion tensor P_{ijk} and hv-curvature tensor P_{hijk} for (λ,β)-metric is given by (25) and (26) respectively.

Now, the T-tensor is given by

$$T_{hijk} = LC_{hij}|_k + l_iC_{hjk} + l_jC_{hik} + l_kC_{hij} + l_hC_{ijk}.$$

Now, the v-derivative of C_{hij} is given by

$$2p_{-(m-2)}C_{hij}|_k = 2p^2_{-(m-2)}a_{hij}|_k +$$
$$2p^2_{-(m-2)}|_k\, a_{hij} - \frac{1}{L}[h_{jk}(l_iP_h + l_hP_i)$$
$$+h_{ik}(l_jP_h + l_hP_j) + h_{hk}(l_iP_j + l_jP_i)] + \quad \text{....................(27)}$$
$$\underset{(hij)}{\Pi}(H_{hi}P_j|_k) + \frac{2r_{-(m-1)}}{3p_{-(m-1)}}\underset{(hij)}{\Pi}(P_hP_iP_j|_k)$$

Using (6), (13b) and (27), the T-tensor for (λ,β)-metric is given by

$$T_{hijk} = \frac{1}{2p_{-(m-2)}}[2p^2_{-(m-2)}La_{hij}|_k +$$
$$2p^2_{-(m-2)}|_k\, La_{hij} - 2p_{-(m-2)}|_k\, LC_{hij}]$$
$$-\frac{1}{2p_{-(m-2)}}[h_{jk}(l_iP_h + l_hP_i) +$$
$$h_{ik}(l_jP_h + l_hP_j) + h_{hk}(l_iP_j + l_jP_i)]$$
$$+L\underset{(hij)}{\Pi}(H_{hi}P_j|_k) + \frac{2r_{-(m-1)}}{3p^3_{-(m-1)}}L\underset{(hij)}{\Pi}(P_hP_iP_j|_k)$$
$$+p^2_{-(m-2)}\underset{(hijk)}{\Pi}(l_ha_{ijk})$$
$$+\underset{(hijk)}{\Pi}\{H_{hi}(l_jP_k + l_kP_j)\}] \quad \text{......................(28)}$$

Thus,

Proposition 7: The T-tensor T_{hijk} for (λ,β)-metric is given by (28).

CONCLUSION

Present paper examined some conditions that characterize The normalized supporting element l_i, angular metric tensor h_{ij}, metric tensor g_{ij} and (h) hv-torsion tensor C_{ijk}, The reciprocal of the metric tensor g_{ij} ,. T-tensor T_{hijk}, C-reducile of Finsler space with (λ,β)-metric,.

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