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The Stability of Solutions of Sitnikov Restricted Problem of three Bodies When the Primaries are Triaxial Rigid Bodies

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ABSTRACT

The paper deals with the stability of the solutions of Sitnikov's restricted problem of three bodies if the primaries are triaxial rigid bodies. The infinitesimal mass is moving in space and is being influenced by motion of two primaries ($m_1 > m_2$). They move in circular orbits without rotation around their centre of mass. Both primaries are considered as axis symmetric bodies with one of the axes as axis of symmetry whose equatorial plane coincides with motion of the plane. The synodic system of co-ordinates initially coincides with inertial system of co-ordinates. It is also supposed that initially the principal axis of the body m_1 is parallel to synodic axis and are of the axes of symmetry is perpendicular to plane of motion.

Keywords-Axis symmetric bodies; equatorial plane; infinitesimal body; Libration points; synodic axes.

INTRODUCTION

The Sitnikov problem is a special case of restricted three body problem .It refers to the motion of the test particle along an axis perpendicular to the plane of motion of two equal primaries that move on elliptic orbits. The axis passes through the center of mass of the system.

The restricted problem of three bodies if primaries are oblate spheroids where equatorial plane coincides with plane of motion and their stability has been studied by Vidyakin (1974). Subba Rao and Sharma (1975) have studied the stability of libration points. EI-Shaboury (1991) also studied the stability of libration points. Khanna and Bhatnagar (2001) have studied the stability of solutions of Sitnikov restricted problem of the smaller primary is a triaxial rigid body.

The stability of motion in the Sitnikov problem have studied by Soulis (2007) and found that as mass of infinitesimal body increases, the domain of allowed motion grows significantly. This paper tries to establish the solutions of Sitnikov restricted problem of these bodies when both the primaries are triaxial rigid body.

Solution by Lindstedt-Poincare Method

Now following Thapa and Hassan (2013)

$$z = c\cos\tau + \frac{c^3}{32}\ \cos\tau - \cos 3\tau\ \left(\frac{\varepsilon}{\eta_0^2}\right)$$

$$+\frac{c^5}{1024}\ 23\cos\tau - 24\cos 3\tau + \cos 5\tau\ \left(\frac{\varepsilon}{\eta_0^2}\right)^2 + \frac{547c^7}{32768}\cos\tau\left(\frac{\varepsilon}{\eta_0^2}\right)^3$$

$$+\frac{c^7}{2048}\left(-\frac{297}{8}\cos 3\tau + 3\cos 5\tau - \frac{\cos 7\tau}{16}\right)\left(\frac{\varepsilon}{\eta_0^2}\right)^3 + \ldots \quad (1)$$

But $$\frac{\varepsilon}{\eta_0^2} = \frac{\frac{48}{a^7}\ a^2 + 5\alpha + 10(\sigma_1 + \sigma_1')}{\frac{8}{a^5}\ a^2 + 3\alpha}$$

$$= \frac{48}{a^7} \times \frac{a^5}{8}\,\frac{a^2 + 5\alpha + 10(\sigma_1 + \sigma_1')}{a^2 + 3\alpha}$$

$$= \frac{6}{a^2}\,\frac{a^2 + 5\alpha + 10(\sigma_1 + \sigma_1')}{a^2\left(1 + \frac{3\alpha}{a^2}\right)}$$

$$= \frac{6}{a^2}\ a^2 + 5\alpha + 10(\sigma_1 + \sigma_1')\ \left(1 + \frac{3\alpha}{a^2}\right)^{-1}$$

$$= \frac{6}{a^4}\ a^2 + 5\alpha + 10(\sigma_1 + \sigma_1')\ \left(1 - \frac{3\alpha}{a^2} + \ldots\right)$$

$$= \frac{6}{a^4}\left(a^2 + 5\alpha + 10(\sigma_1 + \sigma_1') - 3q - \frac{15\alpha^2}{a^2} - \frac{30\alpha}{a^2}(\sigma_1 + \sigma_1')\right)$$

$$= \frac{6}{a^4}(a^2 + 2\alpha) + 10(\sigma_1 + \sigma_1')$$

$$\Rightarrow \left(\frac{\varepsilon}{\eta_0^2}\right)^2 = \frac{36}{a^8}(a^4 + 4a^2\alpha + 4\alpha^2) + 2\cdot\frac{36}{a^8}(a^2 + 2\alpha)\cdot 10(\sigma_1 + \sigma_1')$$

$$= \frac{36}{a^6}(a^2 + 4\alpha) + \frac{720}{a^6}(\sigma_1 + \sigma_1')$$

and $$\left(\frac{\varepsilon}{\eta_0^2}\right)^3 = \left[\frac{6}{a^4}(a^2 + 2\alpha) + 10(\sigma_1 + \sigma_1')\right]\left[\frac{36}{a^6}(a^2 + 4\alpha) + \frac{720}{a^6}(\sigma_1 + \sigma_1')\right]$$

$$= \frac{216}{a^8}(a^2 + 6\alpha) + \frac{720}{a^4}(\sigma_1 + \sigma_1')$$

From (1)

$$z = c\cos\tau + \frac{3c^3}{16a^2}\ \cos\tau - \cos 3\tau\ + \frac{5c^3}{16}\ \cos\tau - \cos 3\tau\ (\sigma_1 + \sigma_1')$$

$$+\frac{c^5}{a^4}\frac{9}{256}\ 23\cos\tau - 24\cos 3\tau + \cos 5\tau$$

$$+\frac{c^5}{a^2}\frac{15}{128}\ 23\cos\tau - 24\cos 3\tau + \cos 5\tau\ (\sigma_1 + \sigma_1')$$

$$+\frac{14769}{4096}\frac{c^7}{a^6}\cos\tau + \frac{393840}{32768}\frac{c^7}{a^4}\cos\tau(\sigma_1 + \sigma_1')$$

$$+\frac{c^7}{a^6}\frac{216}{2048}\left(-\frac{297}{8}\cos 3\tau + 3\cos 5\tau - \frac{1}{16}\cos 7\tau\right)$$

$$+\frac{c^7}{a^4}\cdot\frac{720}{2048}\left(-\frac{297}{8}\cos 3\tau + 3\cos 5\tau - \frac{1}{16}\cos 7\tau\right)(\sigma_1 + \sigma_1')$$

$$+\alpha\left[\frac{c^3}{a^4}\cdot\frac{3}{8}\ (\mathrm{Cos}\,\tau - \cos 3\tau)\ \ \frac{c^5}{a^6}\cdot\frac{144}{1024}\ (2\mathrm{Cos}\ \tau - 24\cos 3\tau + \cos 5\tau)\right.$$

$$\left.+\frac{547\times 216\times 6}{32768}\frac{c^7}{a^8}\cos\tau + \frac{c^7}{a^8}\frac{216\times 6}{2048}\left(-\frac{297}{8}\cos 3\tau + 3\cos 5\tau - \frac{1}{16}\cos 7\tau\right) + \ldots\right]$$

Stability of the motion

The general equations of motion of the infinitesimal mass under the mutual gravitational field of two axis symmetric bodies (triaxial rigid bodies) are

$$\left.\begin{aligned}\ddot{x} - 2x\dot{y} &= \frac{\partial\Omega}{\partial x}\\ \ddot{y} + 2n\dot{x} &= \frac{\partial\Omega}{\partial y}\\ \ddot{z} &= \frac{\partial\Omega}{\partial z}\end{aligned}\right\}$$

When

$$\Omega = \frac{1}{\left(z^2 + \frac{a^2}{4}\right)^{\frac{1}{2}}} + \frac{\alpha}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}}} - \frac{3\ \sigma_1 + \sigma_2'\ z^2}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}}}$$

Where α = $2\sigma_1 - \sigma_2 + 2\sigma_1' - \sigma_2'$

Since Ω is independent of *x* and *y*, but depending upon *z* only

hence, $\frac{\partial\Omega}{\partial x} = 0$

$$\frac{\partial\Omega}{\partial y} = 0$$

$$\frac{\partial\Omega}{\partial z} = -\frac{z}{\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}}} - \frac{3\alpha z}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}}} + \frac{15(\sigma_1 + \sigma_1')z^3}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{7}{2}}}$$

Also $\frac{\partial^2\Omega}{\partial x^2} = 0, \quad \frac{\partial^2\Omega}{\partial x\partial y} = 0, \frac{\partial^2\Omega}{\partial x\partial z} = 0$

$$\frac{\partial^2\Omega}{\partial y^2} = 0, \quad \frac{\partial^2\Omega}{\partial y\partial z} = 0$$

$$\frac{\partial^2\Omega}{\partial z^2} = -\left[\frac{\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}} - z\frac{3}{2}\left(z^2 + \frac{a^2}{4}\right)^{\frac{1}{2}} 2z}{\left(z^2 + \frac{a^2}{4}\right)^{3}}\right]$$

$$-\frac{3\alpha}{4}\left[\frac{\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}} - z\frac{5}{2}\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}} 2z}{\left(z^2 + \frac{a^2}{4}\right)^{5}}\right]$$

$$+\frac{15(\sigma_1 + \sigma_1')}{4}\left[\frac{\left(z^2 + \frac{a^2}{4}\right)^{\frac{7}{2}} 3z^2 - z^3\frac{7}{2}\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}} 2z}{\left(z^2 + \frac{a^2}{4}\right)^{7}}\right]$$

$$= -\frac{1}{\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}}} + \frac{3z^2}{\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}}} - \frac{3\alpha}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}}}$$

$$+\frac{15\alpha z^2}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{7}{2}}} + \frac{45(\sigma_1 + \sigma_1')z^2}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{7}{2}}} - \frac{105(\sigma_1 + \sigma_1')z^4}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{9}{2}}} \quad (2)$$

$$\Omega_{zz} = -\frac{1}{\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}}} + \frac{3\ 4z^2 - \alpha}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}}} + \frac{15z^2\ \alpha + 3\sigma_1 + 3\sigma_1'}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{7}{2}}} - \frac{105(\sigma_1 + \sigma_1')z^4}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{9}{2}}}$$

Let us write the equations as

$$\left.\begin{aligned}&\ddot{x} - 2x\dot{y} = 0 = f(x, y, z)\\ &\ddot{y} + 2n\dot{x} = 0 = g(x, y, z)\\ &\ddot{z} = -\frac{z}{\left(z^2 + \frac{a^2}{4}\right)^{\frac{3}{2}}} - \frac{3\alpha z}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{5}{2}}} + \frac{15(\sigma_1 + \sigma_1')z^3}{4\left(z^2 + \frac{a^2}{4}\right)^{\frac{7}{2}}}\\ &\quad = h(x, y, z).\end{aligned}\right\}$$

For the conditions of stable solution, the square of characteristic roots must be less then or equal to zero.

i.e. $\lambda_i^2 \le 0 \quad$ for $i = 1, 2, 3.$

But $\lambda_1^2 = 0 \Rightarrow \lambda_{11} = \lambda_{12} = 0.$

$\lambda_2^2 = -4n^2 \Rightarrow \lambda_{21} = 2ni, \lambda_{22} = -2ni.$

When $\lambda_3^2 = \Omega_{zz}^0.$

Now,

$$\Omega_{zz}^{0}=-\frac{1}{\left(z_0^2+\frac{a^2}{4}\right)^{\frac{3}{2}}}+\frac{3\ \ 4z_0^2-\alpha}{4\left(z_0^2+\frac{a^2}{4}\right)^{\frac{5}{2}}}+\frac{15}{4}z_0^2\frac{\alpha+3\sigma_1+3\sigma_1'}{\left(z_0^2+\frac{a^2}{4}\right)^{\frac{7}{2}}}-\frac{105}{4}\frac{\sigma_1+\sigma_1'\ \ z_0^4}{\left(z_0^2+\frac{a^2}{4}\right)^{\frac{9}{2}}}$$

$$=-\frac{1}{\left(\frac{a}{2}\right)^3\left(1+\frac{4z_0^2}{a^2}\right)^{\frac{3}{2}}}+\frac{3}{4}\frac{4z_0^2-\alpha}{\left(\frac{a}{2}\right)^5\left(1+\frac{4z_0^2}{a^2}\right)^{\frac{5}{2}}}+\frac{15}{4}z_0^2\frac{\alpha+3\sigma_1+3\sigma_1'}{\left(\frac{a}{2}\right)^7\left(1+\frac{4z_0^2}{a^2}\right)^{\frac{7}{2}}}$$

$$-\frac{105}{4}\frac{(\sigma_1+\sigma_1'\ z_0^4)}{\left(\frac{a}{2}\right)^9\left(1+\frac{4z_0^2}{a^2}\right)^{\frac{9}{2}}}$$

$$=-\frac{8}{a^3}\left(1+\frac{4z_0^2}{a^2}\right)^{-\frac{3}{2}}+\frac{3}{4}\times\frac{32}{a^5}\ \ 4z_o^2-\alpha\ \ \left(1+\frac{4z_0^2}{a^2}\right)^{-\frac{5}{2}}$$

$$+\frac{15}{4}z_0^2\times\frac{128}{a^7}\ \ \alpha+3\sigma_1+3\sigma_1'\ \ \left(1+\frac{4z_0^2}{a^2}\right)^{-\frac{7}{2}}$$

$$-\frac{105}{4}\times\frac{2^9}{a^9}\ \ \sigma_1+\sigma_1'\ \ z_0^4\left(1+\frac{4z_0^2}{a^2}\right)^{-\frac{9}{2}}$$

$$=-\frac{8}{a^3}+\frac{48}{a^3}\left(\frac{z_0}{a}\right)^2+\frac{24}{a^3}\left[4\left(\frac{z_0}{a}\right)^2-\frac{\alpha}{a^2}\right]\left[1-10\left(\frac{z_0}{a}\right)^2+\ldots\right]$$

$$+\frac{15\cdot 32}{a^5}\left(\frac{z_0}{a}\right)^2\ \ \alpha+3\sigma_1+3\sigma_1'\ \ \left[1-14\left(\frac{z_0}{a}\right)^2+\ldots\right]$$

$$-\frac{105\times 128\ \ \sigma_1+\sigma_1'}{a^5}\left(\frac{z_0}{a}\right)^4\left[1-18\left(\frac{z_0}{a}\right)^2+\ldots\right]$$

$$=-\frac{8}{a^3}+\frac{48}{a^3}\left(\frac{z_0}{a}\right)^2+\frac{24}{a^3}\left[4\left(\frac{z_0}{a}\right)^2-\frac{\alpha}{a^2}+\frac{10\alpha}{a^2}\left(\frac{z_0}{a}\right)^2-\ldots\right]$$

$$+\frac{480}{a^5}\left(\frac{z_0}{a}\right)^2\ \ \alpha+3\sigma_1+3\sigma_1'\ \ \left(\frac{z_0}{a}\right)^2+\ldots$$

$$=-\frac{8}{a^3}+\frac{48}{a^3}\left(\frac{z_0}{a}\right)^2+\frac{96}{a^3}\left(\frac{z_o}{a}\right)^2-\frac{24\alpha}{a^5}+\frac{480}{a^5}\ \ 5\sigma_1+5\sigma_1'-\sigma_2-\sigma_2'\ \ \left(\frac{z_0}{a}\right)^2+\ldots$$

Neglecting higher order infinitesimal above the second order

$$\Omega_{zz}^{0}=-\frac{8}{a^3}+\frac{144}{a^3}\left(\frac{z_0}{a}\right)^2-\frac{24\alpha}{a^5}$$

$$=-\frac{1}{a^5}\left[8a^2-144z_0^2+24\alpha\right]$$

$$=-\frac{8}{a^5}\left[a^2-18z_0^2+3\alpha\right]$$

$$=-\frac{8}{a^5}\left[a^2-18z_0^2+3\ \ 2\sigma_1-\sigma_2+2\sigma_1'-\sigma_2'\ \ \right]$$

Hence if $a^2-18z_0^2+3\alpha>0$ then $\Omega_{zz}^0<0$ and $\lambda_3^2<0$, λ_{31} & λ_{32} both are imaginary. Thus all the characteristic roots of the coefficient matrix A are either zero or imaginary. Hence the matrix of Sitnikov is stable.

When $a^2-18z_0^2+3\ \ 2\sigma_1-\sigma_2+2\sigma_1'-\sigma_2'\ \ >0$

$$\Rightarrow z_0^2<\frac{a^2+3\ \ 2\sigma_1-\sigma_2+2\sigma_1'-\sigma_2'}{18}$$

$$\Rightarrow-\sqrt{\frac{a^2+3\ \ 2\sigma_1-\sigma_2+2\sigma_1'-\sigma_2'}{18}}<z_0<\sqrt{\frac{a^2+3\ \ 2\sigma_1-\sigma_2+2\sigma_1'-\sigma_2'}{18}}$$

Thus the infinitesimal mass will perform oscillatory motion from $-\sqrt{\frac{a^2+3\alpha}{18}}$ to $\sqrt{\frac{a^2+3\alpha}{18}}$ about the centre of mass 0.

CONCLUSION

If the primaries are triaxial rigid bodies then the motion of the third body depends on parameter τ, distance ‘a’ between the primaries, constant c and $\alpha=2\sigma_1-\sigma_2+2\sigma_1'-\sigma_2'$. In this case, the infinitesimal mass m_3 will perform oscillatory motion in between $-\sqrt{\frac{a^2+3\alpha}{18}}$ to $\sqrt{\frac{a^2+3\alpha}{18}}$, so that the condition of stability are satisfied by the solution of the Sitnikov motion.

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