

Sequencing Time Window for the Total Product Rate Variation Problem

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ABSTRACT

The problem of minimizing the total deviations between the actual and the ideal cumulative production of a variety of models of a common base product arises as a sequencing problem in mixed-model just-in-time production systems. This is called the total product rate variation problem. Several pseudo-polynomial exact algorithms and heuristics have been derived for this problem. In this paper, we estimate the sequencing time window for all copies of all models of a common base product for the problem when the m -th power of the total deviations between the actual and the ideal cumulative productions has to be minimized.

Key words: Bound, product rate variation problem, non-linear integer programming problem.

INTRODUCTION

Many companies have changed the assembly lines from paced single-model lines for mass production to mixed-model assembly lines for mass customization of a variety of models of a common base product. Just-in-time production system which requires producing only the necessary products in the necessary quantities at the necessary times usually uses mixed-model assembly lines.

Mixed-model just-in-time production systems with negligible change-over costs between the models have been used in order to respond to the customer demands for a variety of models of a common base product without holding large inventories or incurring large shortages. One of the most important problems for the effective utilization of the systems consists in sequencing different models with keeping the rate of usage of all parts used by the assembly lines as constant as possible. The problem is known as the mixed-model just-in-time sequencing problem (abbreviated as MMJITSP). The problem of minimizing the variation in the rate at which different models are produced on the line is called the product rate variation problem (abbreviated as PRVP). The latter problem is the single-level case of MMJITSP. The problem of minimizing the total deviations between the actual cumulative productions from the ideal one is called the total PRVP (abbreviated as TPRVP), see Kubiak (1993). This problem has been widely investigated in the literature since it has a model with a strong mathematical base and wide real-world applications, see Dhamala & Khadka (2009), a recent survey.

In this paper, we determine the sequencing time window for all copies of all models of a common base product. The remainder of the paper is as follows. In the second section, we present a mathematical modeling of the problem. In the third section, we present the sequencing time window for all copies of all models of a common base product which is the major contribution of this paper.

First, the level curves are investigated, then the largest function value and finally the smallest function value. The last section concludes the paper.

MATHEMATICAL MODELLING

Let D be the total demand of n different models with d_i copies of model $i, i = 1, 2, \dots, n$, where $n \geq 2$ and $D = \sum_{i=1}^n d_i$. The time horizon is partitioned into D equal time units under the assumption that each copy of a model $i, i = 1, \dots, n$, has equal processing time. A copy of a model is produced in a time unit $k, k = 1, \dots, D$, means that the copy of the model is produced during the time period from $k - 1$ to k . Let $r_i = \frac{d_i}{D}$ be the demand rate. Let x_{ik} and kr_i be the actual and the ideal cumulative productions, respectively, of model i produced during the time units 1 through k . An inventory holds if $x_{ik} - kr_i > 0$, and a shortage incurs if $kr_i - x_{ik} > 0$. We assign the same cost for both inventory and shortage. Miltenburg (1989) and Kubiak and Sethi (1991, 1994) gave an integer programming formulation for TPRVP as follows

with m being a positive integer:

$$\text{minimize } [F_m = \sum_{k=1}^D \sum_{i=1}^n |x_{ik} - kr_i|^m]$$

subject to

$$\begin{aligned} \sum_{i=1}^n x_{ik} &= k, & k &= 1, 2, \dots, D \\ x_{i(k-1)} &\leq x_{ik}, & i &= 1, 2, \dots, n; k = 2, 3, \dots, D \\ x_{iD} &= d_i, x_{i0} = 0, & i &= 1, 2, \dots, n \\ x_{ik} &\geq 0, \text{ integer } i &= 1, 2, \dots, n; k &= 1, 2, \dots, D. \end{aligned}$$

LEVEL CURVE AND BOUNDS

There exist nD deviations between the actual and the ideal cumulative productions of D copies of n models. The value of the actual cumulative production x_{ik} , $i = 1, 2, \dots, n; k = 1, 2, \dots, D$, is sequence-dependent integer from $\{0, 1, \dots, d_i\}$. However, the value of the ideal cumulative production kr_i , $i = 1, 2, \dots, n; k = 1, 2, \dots, D$, is sequence-independent rational number. Let j be the number of copies of a model and (i, j) be the j^{th} copy of model i , $i = 1, 2, \dots, n$. The actual cumulative production x_{ik} , $i = 1, 2, \dots, n; k = 1, 2, \dots, D$, has nD values with $x_{ik} \in \{j | j = 0, 1, 2, \dots, d_i; i = 1, 2, \dots, n\}$. There exist at most $n + D$ different values of x_{ik} for TPRVP. Hence, one can replace x_{ik} by j with $j = 0, 1, \dots, d_i; i = 1, 2, \dots, n$, in the level curve of the objective value of the function of TPRVP. The level curve for copy (i, j) of the objective function of TPRVP is defined as

$$f_{ij}^m = nD|j - kr_i|^m, i = 1, 2, \dots, n; j = 0, 1, \dots, d_i.$$

A perfect matching of copies and time units relies on the level curves and the bound $B > 0$ of the function F_m of PRVP that are drawn over the planning horizon. The points at which the bound intersects the level curves are useful to find the sequencing times. A copy (i, j) is sequenced at a time unit $k \in \{1, \dots, D\}$ such that the level curve does not exceed the bound B . An upper bound on the absolute deviation objective function $|x_{ik} - kr_i|$ for BPRVP is $UB^*_1 = 1$, see Steiner and Yeomans (1993), and a better one has been given as $UB^*_1 = 1 - \frac{1}{D}$, see Brauner and Crama (2004). Since the points, where the bound $UB^*_1 = 1$ intersects the level curves of BPRVP with the objective function $|x_{ik} - kr_i|^m$ for different values of m , are the same, the bound $UB^*_1 = 1$ is also an upper bound for BPRVP with the objective function $|x_{ik} - kr_i|^m$ for all values of m . However, the upper bounds corresponding to $UB^*_1 = 1 - \frac{1}{D}$ are different for BPRVP with the objective functions $|x_{ik} - kr_i|^m$ for different values of m . An upper bound on the largest value of the objective function $|x_{ik} - kr_i|^m$ of BPRVP has been established as

$$UB^*_m = \left(1 - \frac{1}{D}\right)^m,$$

Dhamala *et al.* (2010) and Khadka (2012). A minor modification yields an upper bound on the optimal value of function F_m for PRVP given by

$UB_m = nD\left(1 - \frac{1}{D}\right)^m$, Khadka (2014) (although it not tight). The lower bound $LB^*_1 = 1 - r_{max}$ on the absolute deviation objective function $|x_{ik} - kr_i|$ for BPRVP has been established by Steiner and Yeomans (1993), and it has been modified as

$LB^*_m = (1 - r_{max})^m$, for this problem with the objective function $|x_{ik} - kr_i|^m$ (Dhamala *et al.* (2010) and Khadka (2012)). Hence, a lower bound on the optimal value of the objective function F_m for TPRVP is given by

$$LB_m = nD(1 - r_{max})^m, \text{ Khadka (2014).}$$

SEQUENCING TIME WINDOW

The earliest and the latest sequencing times are determined by the level curve and a suitably chosen bound. The selection of an upper bound always yields the sequencing times that give rise to a feasible solution. However, this is time-consuming. A selection of a lower bound for small-size instances is better for BPRVP (see Kovalyov *et al.* (2001)). A feasible solution corresponding to the lower bound is optimal.

Let $E_m(i, j)$ and $L_m(i, j)$, be the earliest and the latest sequencing times of copy (i, j) for a given (suitably chosen) bound B , respectively, when considering the objective function F_m .

$$\text{The integer time interval } T_{(i,j)m} = [E_m(i, j), L_m(i, j)]$$

is called the sequencing time window for the copy (i, j) since the level curve of the objective function F_m of TPRVP does not exceed the bound B if copy (i, j) is sequenced within this window. The earliest sequencing time $E_m(i, j) \in \{1, \dots, D\}$ is the unique integer such that, when copy (i, j) is sequenced at time $E_m(i, j) - 1$, the level curve exceeds the bound B but does not exceed this bound when sequenced at $E_m(i, j)$.

Similarly, the latest sequencing time $L_m(i, j) \in \{1, \dots, D\}$ is the unique integer such that, when copy $(i, (j - 1))$ is sequenced at time $L_m(i, j) - 1$, the level curve does not exceed the bound B but it exceeds this bound when sequenced at time $L_m(i, j)$.

Theorem 1. Let $E_m(i, j)$ and $L_m(i, j)$, $i = 1, \dots, n; j = 0, 1, \dots, d_i$, be the earliest and the latest sequencing times, respectively, for a given (suitably chosen) bound $B > 0$ for an instance (d_1, \dots, d_n) . Then the equalities

$$E_m(i, j) = \left\lceil \frac{j - \frac{m\sqrt{B}}{\sqrt{nD}}}{r_i} \right\rceil \text{ and } L_m(i, j) = \left\lfloor \frac{j - 1 + \frac{m\sqrt{B}}{\sqrt{nD} + 1}}{r_i} \right\rfloor \text{ hold.}$$

Proof. Let $E_m(i, j)$ and $L_m(i, j)$, $i = 1, \dots, n$; $j = 0, 1, \dots, d_i$, be the earliest and the latest sequencing times, respectively.

The time $E_m(i, j)$ satisfies the two inequalities

$$nD [j - (E_m(i, j) - 1)r_i]^m > B \text{ and} \\ nD [j - E_m(i, j) r_i]^m \leq B$$

This implies the inequalities

$$1 + \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} > E_m(i, j) \text{ and } \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \leq E_m(i, j)$$

Therefore, we have the inequalities

$$\frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \leq E_m(i, j) < 1 + \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i}$$

Hence, we obtain $E_m(i, j) = \left\lceil \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \right\rceil$

Similarly, the time $L_m(i, j)$ satisfies the inequalities

$$nD [(L_m(i, j) - 1)r_i - (j - 1)]^m \leq B \text{ and} \\ nD [L_m(i, j) r_i - (j - 1)]^m > B$$

This implies the inequalities

$$1 + \frac{j^{-1+m} \sqrt{\frac{B}{nD}}}{r_i} \geq L_m(i, j) \text{ and } \frac{j^{-1+m} \sqrt{\frac{B}{nD}}}{r_i} < L_m(i, j)$$

We have $\frac{j^{-1+m} \sqrt{\frac{B}{nD}}}{r_i} < L_m(i, j) \leq 1 + \frac{j^{-1+m} \sqrt{\frac{B}{nD}}}{r_i}$

Thus, we obtain $L_m(i, j) = \left\lfloor \frac{j^{-1+m} \sqrt{\frac{B}{nD}}}{r_i} + 1 \right\rfloor$

Hence the result.

Corollary 1 Let $E_m(i, j)$ and $L_m(i, j)$, $i = 1, \dots, n$; $j = 1, \dots, d_i$, be the earliest and the latest sequencing times, respectively. Then the inequality $E_m(i, j) \leq L_m(i, j)$ holds.

Proof. Consider the earliest sequencing time

$$E_m(i, j) = \left\lceil \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \right\rceil$$

Without loss of optimality, one can use the bound $B = UB_m$, i.e., $B = nD(1 - \frac{1}{D})^m$.

Thus, we obtain

$$E_m(i, j) = \left\lceil \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \right\rceil = \left\lceil \frac{j^{-1+(1-\frac{1}{D})}}{r_i} + \frac{1+2(\frac{1}{D}-1)}{r_i} \right\rceil \\ \leq \left\lfloor \frac{j-1+(1-\frac{1}{D})}{r_i} + 1 + \frac{1+2(\frac{1}{D}-1)}{r_i} \right\rfloor$$

Since $\frac{1+2(\frac{1}{D}-1)}{r_i} \leq 0$ for any $D \geq 2$,

we can write

$$E_m(i, j) \leq \left\lfloor \frac{j^{-1+(1-\frac{1}{D})}}{r_i} + 1 \right\rfloor = \left\lfloor \frac{j^{-1+\frac{m\sqrt{\frac{B}{nD}}}}{r_i} + 1 \right\rfloor = L_m(i, j)$$

Corollary 2 Let $[1, D]$ be the planning horizon and

$$T_{(i,j)m}, i = 1, \dots, n; j = 1, \dots, d_i,$$

be the sequencing time window for copy (i, j) . Then the inclusion $T_{(i,j)m} \subseteq [1, D]$

holds for $i = 1, \dots, n; j = 1, \dots, d_i$.

Proof. Suppose that copy (i, j) , $i = 1, \dots, n; j = 1, \dots, d_i$, is sequenced at time $k \in T_{(i,j)m}$.

Thus, $E_m(i, j) \leq k \leq L_m(i, j)$.

The minimum value of the earliest sequencing time $E_m(i, j)$ for a copy (i, j) can be the first time unit 1. Likewise, the maximum value of the latest sequencing time $L_m(i, j)$ for a copy (i, j) can be the last time unit D . i.e., $1 \leq E_m(i, j)$ and $L_m(i, j) \leq D$.

Thus, $1 \leq k \leq D$.

Hence, the inclusion $T_{(i,j)m} \subseteq [1, D]$ holds.

Corollary 3 Let $[1, D]$ be the planning horizon and

$$T_{(i,j)m}, i = 1, \dots, n; j = 1, \dots, d_i,$$

be the sequencing time window for copy (i, j) . Then

$$\cup^{n.d_i}_{i,j=1} T_{(i,j)m} = [1, D].$$

Proof. Since, by using Corollary 2, $T_{(i,j)m} \subseteq [1, D]$

for each copy (i, j) , $i = 1, \dots, n; j = 1, \dots, d_i$, we have $\cup^{n.d_i}_{i,j=1} T_{(i,j)m} \subseteq [1, D]$

The planning horizon is partitioned into exactly D equal time units, where D is equal to the total number of all copies $j, j = 1, \dots, d_i$, of all models $i, i = 1, \dots, n$. There exists at least one sequencing time window that contains the first time unit 1 and at least another one that contains the last time unit D . This implies $[1, D] \subseteq \cup^{n.d_i}_{i,j=1} T_{(i,j)m}$

Corollary 4 Given a suitably chosen bound $B > 0$, the level curve of the objective function F_m does not exceed B if and only if copy (i, j) is sequenced within the sequencing time window

$$T_{(i,j)m}, i = 1, \dots, n; j = 1, \dots, d_i.$$

Proof. If the level curve

$$f^m_{ij} = nD|j - kr_i|^m$$

of the objective function F_m of TPRVP does not exceed a suitably chosen bound $B > 0$, by Theorem 1, copy (i, j) is sequenced at a time unit $k \in T_{(i,j)m} \subseteq [1, D]$.

Conversely, let copy (i, j) be sequenced at a time unit

$$k \in T_{(i,j)m} \subseteq [1, D].$$

i.e., inequalities

$$E_m(i, j) \leq k \leq L_m(i, j)$$

hold. Thus,

$$\left\lfloor \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \right\rfloor \leq k \text{ and}$$

$$k \leq \left\lceil \frac{j^{-1+m} \sqrt{\frac{B}{nD}} + 1}{r_i} \right\rceil$$

which yields the inequalities

$$\frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \leq k \text{ and } k \leq \frac{j^{-1+m} \sqrt{\frac{B}{nD}}}{r_i} + 1.$$

Thus, we get $nD |j - kr_i|^m \leq B$ and

$$nD |(k - 1)r_i - (j - 1)|^m \leq B.$$

CONCLUSION

For the total product rate variation problem, several pseudo-polynomial exact solution algorithms and heuristics have been developed. The earliest and the latest sequencing times of a feasible solution for TPRVP are

$$E_m(i, j) = \left\lfloor \frac{j^{-m} \sqrt{\frac{B}{nD}}}{r_i} \right\rfloor \text{ and } L_m(i, j) = \left\lceil \frac{j^{-1+m} \sqrt{\frac{B}{nD}} + 1}{r_i} \right\rceil$$

respectively. These sequencing times can be used to develop an $O(D \log D)$ exact solution procedure recently given by Khadka and Werner (2014) which improves the known exact algorithm by Kubiak from (1993) with a complexity of $O(D^3)$.

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