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On Locally Convex Topological Vector Space Valued Null Function Space c_0 (S,T, Φ, ξ, u) Defined by Semi Norm and Orlicz Function

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ABSTRACT

The aim of this paper is to introduce and study a new class c_0 (S, T, Φ, ξ, u) of locally convex space T- valued functions using Orlicz function Φ as a generalization of some of the well known sequence spaces and function spaces. Besides the investigation pertaining to the structures of the class c_0 (S, T, Φ, ξ, u), our primarily interest is to explore some of the containment relations of the class c_0 (S, T, Φ, ξ, u) in terms of different ξ and u so that such a class of functions is contained in or equal to another class of similar nature.

Keywords: Solid space, Orlicz function, Orlicz space, locally convex topological vector space, seminorm

INTRODUCTION

We begin with recalling some notations and basic definitions that are used in this paper.

Definition 1: A sequence space S is said to be solid if $\bar{\xi} = \langle \xi_k \rangle \in S$ and $\bar{\gamma} = \langle \gamma_k \rangle$ a sequence of scalars with $|\gamma_k| \leq 1$, for all $k \geq 1$, then $\bar{\gamma}\,\bar{\xi} = \langle \gamma_k\, \xi_k \rangle \in S$.

So far, a good number of research works have been done on various types of sequence spaces and function spaces.

Definition 2: By an Orlicz function we mean a continuous, non decreasing and convex function

Φ: $[0,\infty) \rightarrow [0,\infty)$ satisfying

$\Phi(0) = 0$, $\Phi(s) > 0$ for $s > 0$ and $\Phi(s) \rightarrow \infty$ as $s \rightarrow \infty$.

It is noted that an Orlicz function is always unbounded and an Orlicz function satisfies the inequality

$\Phi(\gamma s) \leq \gamma\, \Phi(s)$, $0 < \gamma < 1$ (Krasnosel'skiî & Rutickiî, 1961).

An Orlicz function Φ can be represented in the following integral form

$$\Phi(\xi) = \int_0^{\xi} q(t)\, dt$$

where q, known as the kernel of Φ, is right-differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non decreasing,and $q(t) \rightarrow \infty$ as $t \rightarrow \infty$ (Krasnosel'skiî & Rutickiî , 1961).

Definition 3: Lindenstrauss and Tzafriri (1971) used the notion of Orlicz function to construct the sequence space l_Φ of scalars $\langle \xi_k \rangle$ such that $\sum_{k=1}^{\infty} \Phi\left(\frac{\|\xi_k\|}{r}\right) < \infty$ for some $r > 0$. They proved that the space l_Φ equipped with the norm defined by

$$\|\bar{\xi}\|_\Phi = \inf\left\{ r > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|\xi_k|}{r}\right) \leq 1 \right\}$$

This becomes a Banach space, which is called an *Orlicz sequence space*. The space l_Φ (S) is closely related to the space l_p which is an Orlicz sequence space with

$\Phi(s) = s^p : 1 \leq p < \infty$.

Subsequently, Kamthan and Gupta (1981), Rao and Ren (1991), Parashar and Choudhary (1994), Chen (1996), Ghosh and Srivastava (1999), Rao and Subremanina (2004), Savas and Patterson (2005), Bhardwaj and Bala (2007), Khan (2008), Basariv and Altundag (2009), Kolk (2011), Pahari (2013), Srivastava and Pahari (2011 & 2013) and many others have been introduced and studied the algebraic and topological properties of various sequence spaces using Orlicz function as a generalization of several well known sequence spaces.

Definition 4: A topological linear space S is a vector space S over a topological field K (most often the complex numbers $\boldsymbol{C}$ with their standard topologies) which is endowed with a topology $\Im$ such that if s_1, s_2 $\in S$, $\alpha \in$ **K**; the mappings:

(i) vector addition $S \times S \rightarrow S$ such that $(s_1, s_2) \rightarrow s_1 + s_2$ and (ii) scalar multiplication $\mathbf{K} \times S \rightarrow S$ such that $(\alpha, s) \rightarrow \alpha s$ are continuous.

This topology $\mathcal{J}$ is called a *vector topology* or a *linear topology* on *S*. If $\mathcal{J}$ is given by some metric then the topological vector space is called a linear metric space. All normed spaces or inner product spaces endowed with the topology defined by its norm or inner product are well-known examples of topological vector spaces.

A local base of topological vector space *S* is a collection B of neighbourhood θ such that every neighbourhood of θ contains the member of B.

A set *S* in a topological vector space *S* is said to be absorbing if for every $s \in S$ there exists an $\alpha > 0$ such that $s \in \nu S$ for all $\nu \in \mathbf{C}$ such that $|\nu| \geq \alpha$; and *balanced* if $\nu S \subset S$ for every $\nu \in \mathbf{C}$ such that $|\nu| \leq 1$.

It is called *convex* in *S* if for every $\alpha \geq 0$, we have

$\alpha S + (1-\alpha) S \subset S$; and absolutely convex in *S* if it is both balanced and convex.

Definition 5: The *gauge* or Minkowski functional of a set *A* in a vector space *X* is a map $x \rightarrow q_A(x)$ from *X* into the extended set $\boldsymbol{R}_+ \cup \{\infty\}$ of non-negative real numbers defined as follows:

$$q_A(x) = \begin{cases} \inf\ r, & \text{if there exists } r > 0 \text{ such that } x \in rA \text{ and} \\ \infty, & \text{if } x \notin rA \text{ for all } r > 0. \end{cases}$$

Definition 6: A seminorm (pseudonorm) on a linear space *S* over the scalar $\boldsymbol{C}$ with zero element θ is a subadditive function $p : S \rightarrow \boldsymbol{R}_+$ satisfying

$p(\alpha s) = |\alpha|\, p(s)$, for all $\alpha \in \boldsymbol{C}$ and $s \in S$.

Clearly, if $p(s) = 0$ implies $s = \theta$,then *p* is a nom.

If *S* is a vector space equipped with a family $\{p_i : i \in I\}$ of seminorms then there exists a unique locally convex topology $\mathcal{J}$ on *S* such that each p_i is $\mathcal{J}$ - continuous (Rudin, 1991 & Park, 2005).

The class $c_0(S, T, \Phi, \xi, u)$ of locally convex space valued functions

Let *S* be an arbitrary non empty set (not necessarily countable) and $F(S)$ be the collection of all finite subsets of *S*. Let $(T, \Im)$ be a Hausdorff locally convex topological vector space (lcTVS) over the field of complex numbers *C* and T^* be the topological dual of *T*. Let $U(T)$ denotes the fundamental system of balanced, convex and observing neighbourhoods of zero vector θ of *T*. p_U will denote gauge or Minkowski functional of $U \in \mathcal{U}(T)$.

Thus, $D = \{p_U : U \in \mathcal{U}(T)\}$is the collection of all continuous seminorms generating the topology $\Im$ of *T*. Let *u* and *w* be any functions on $S \rightarrow \boldsymbol{R}^+$, the set of positive real numbers, and $l_\infty(S, \boldsymbol{R}^+) = \{ u : S \rightarrow \boldsymbol{R}^+$ such that $\sup_s u(s) < \infty\}$.

Further, we write ξ, η for functions on $S \rightarrow \boldsymbol{C} - \{0\}$, and the collection of all such functions will be denoted by $s(S, \boldsymbol{C} - \{0\})$.

We introduce the following new class of locally convex topological vector space valued functions:

$c_0(S, T, \Phi, \xi, u) = \{\phi : S \rightarrow T$: for every $\varepsilon > 0$ and

$p_U \in D$, there exists $J \in \mathcal{F}(S)$ such that for some $r > 0$,

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) < \varepsilon \text{ for each } s \in S - J\};$$

When $\xi : S \rightarrow \boldsymbol{C} - \{0\}$ is a function such that $\xi(s) = 1$ for all *s*. Then, $c_0(S, T, \Phi, \xi, u)$ will be denoted by $c_0(S,T,\Phi,u)$ and when $u : S \rightarrow \boldsymbol{R}^+$ is a function such that $u(s) = 1$ for all *s*, then $c_0(S, T, \Phi, \xi, u)$ will be denoted by $c_0(S, T, \Phi, \xi)$.

In fact, these classes are the generalizations of the familiar sequence and function spaces, studied in Srivastava (1996), Tiwari *et al*. (2008 & 2010), Pahari (2011), Srivastava and Pahari (2011) using norm.

RESULTS

We explore the structure of the class $c_0(S, T, \Phi, \xi, u)$ of *lc* TVS *T* – valued functions by investigating the conditions in terms of different *u* and ξ so that a class is contained in or equal to another similar class and thereby derive the conditions of their equality.

We shall denote the zero element of this class by θ,which we shall mean the function of $\theta : S \rightarrow T$ such that $\theta(s) = 0$, for all $s \in S$.

Moreover, we shall frequently use the notations

$L = \sup_s u(s)$ and $A[\alpha] = \max(1, |\alpha|)$, for scalar α. But when the functions $u(s)$ and $w(s)$ occur, then to distinguish *L*, we use the notations $L(u)$ and $L(w)$ respectively.

Theorem 1: The class $c_0(S, T, \Phi, \xi, u)$ forms a solid.

Proof:

Let $\phi \in c_0(S, T, \Phi, \xi, u)$, $r > 0$ be associated with ϕ and $\varepsilon > 0$. Then for $p_U \in D$, there exists a $J \in \mathcal{F}(S)$ such that

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) < \varepsilon \text{ for every } s \in S - J.$$

Now, if we take scalars $\alpha(s)$, $s \in S$ such that $|\alpha(s)| \le 1$, then

$$\Phi\left(\frac{[p_U(\alpha(s)\,\xi(s)\,\phi(s))]^{u(s)}}{r}\right) \le \Phi\left(\frac{|\alpha(s)|^{u(s)}\,[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) \le \Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) < \varepsilon.$$

This shows that $\alpha\,\phi \in c_0(S, T, \Phi, \xi, u)$ and hence $c_0(S, T, \Phi, \xi, u)$ is solid.

Theorem 2: If $u \in l_\infty(S, \boldsymbol{R}^+)$ and $\xi, \eta \in s(S, \boldsymbol{C} - \{0\})$, then

$c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \eta, u)$

if $\liminf_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} > 0$.

Proof:

Assume that $\liminf_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} > 0$.

Then there exists $m > 0$ such that,

$m\,|\eta(s)|^{u(s)} < |\xi(s)|^{u(s)}$ for all but finitely many $s \in S$.

Let $\phi \in c_0(S, T, \Phi, \xi, u)$, $r_1 > 0$ be associated with ϕ and $\varepsilon > 0$. Then for $p_U \in D$, there exists $J \in \mathcal{F}(S)$ such that

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r_1}\right) < \varepsilon \text{ for each } s \in S - J.$$

Let us choose r such that $r_1 < m\,r$. Then for such r, using non decreasing property of Φ, we have

$$\Phi\left(\frac{[p_U(\eta(s)\,\phi(s))]^{u(s)}}{r}\right) = \Phi\left(\frac{[|\eta(s)|\,p_U(\phi(s))]^{u(s)}}{r}\right) \le \Phi\left(\frac{[|\xi(s)|\,p_U(\phi(s))]^{u(s)}}{m\,r}\right) \le \Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r_1}\right) < \varepsilon, \text{ for each } s \in S - J.$$

Since $p_U \in D$ is arbitrary in the above discussion, therefore we easily get $\phi \in c_0(S, T, \Phi, \eta, u)$.

This proves that $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \eta, u)$.

Theorem 3: If $u \in l_\infty(S, \boldsymbol{R}^+)$, $\xi, \eta \in s(S, \boldsymbol{C} - \{0\})$ and $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \eta, u)$, then

$\liminf_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} > 0$.

Proof:

Assume that $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \eta, u)$ but $\liminf_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} = 0$.

Then we can find a sequence $< s_k >$ of distinct points in S such that for every $k \ge 1$,

$$k|\xi(s_k)|^{u(s_k)} < |\eta(s_k)|^{u(s_k)}. \quad \text{.................................. (1)}$$

We now choose $t \in T$ and $p_V \in D$ such that $p_V(t) = 1$ and define $\phi : S \to T$ by

$$\phi(s) = \begin{cases} (\xi(s))^{-1}\,k^{-1/u(s)}\,t, & \text{for } s = s_k,\ k = 1, 2, \dots, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \quad \text{.................................. (2)}$$

Let $r > 0$. Then for each $p_U \in D$ and each $k \ge 1$, we have

$$\Phi\left(\frac{[p_U(\xi(s_k)\,\phi(s_k))]^{u(s_k)}}{r}\right) = \Phi\left(\frac{[p_U(k^{-1/u(s_k)}\,t)]^{u(s_k)}}{r}\right) = \Phi\left(\frac{\frac{1}{k}[p_U(t)]^{u(s_k)}}{r}\right) \le \frac{1}{k}\Phi\left(\frac{A\,[(p_U(t))^{L(u)}]}{r}\right) \to 0, \text{as } k \to \infty \text{ and}$$

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) = 0, \text{ for } s \ne s_k,\ k \ge 1.$$

Thus for a given $\varepsilon > 0$, we can find a finite subset J of S satisfying

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) < \varepsilon \text{ for all } s \in S - J.$$

This clearly shows that $\phi \in c_0(S, T, \Phi, \xi, u)$.

But for each $k \ge 1$, in view of equations (1) and (2), we have

$$\Phi\left(\frac{[p_V(\eta(s_k)\,\phi(s_k))]^{u(s_k)}}{r}\right) = \Phi\left(\frac{[p_V(\eta(s_k)\,(\xi(s_k))^{-1}\,k^{-1/u(s_k)}\,t)]^{u(s_k)}}{r}\right) = \Phi\left(\frac{1}{k\,r}\left|\frac{\eta(s_k)}{\lambda(s_k)}\right|^{u(s_k)}[p_V(t)]^{u(s_k)}\right) \ge \Phi\left(\frac{1}{r}\right),$$

which is independent of k.

This shows that $\phi \notin c_0(S, T, \Phi, \eta, u)$, a contradiction. This completes the proof.

On combining the Theorems 2 and 3, we get:

Theorem 4: If $u \in l_\infty(S, R^+)$ and $\xi, \eta \in s(S, C - \{0\})$, then $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \eta, u)$

if and only if $\liminf_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} > 0$.

Theorem 5: Let $u \in l_\infty(S, R^+)$. Then for any $\xi, \eta \in s(S, C - \{0\})$, $c_0(S, T, \Phi, \eta, u) \subset c_0(S, T, \Phi, \xi, u)$,

if $\limsup_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} < \infty$.

Proof:

Assume that $\limsup_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} < \infty$.

Then there exists a constant $d > 0$ such that

$|\xi(s)|^{u(s)} < d\,|\eta(s)|^{u(s)}$ for all but finitely many $s \in S$.

Let $\phi \in c_0(S, T, \Phi, \eta, u)$, $r_1 > 0$ be associated with ϕ and $\varepsilon > 0$. Then for $p_U \in D$, there exists $J \in \mathcal{F}(S)$ such that

$$\Phi\left(\frac{[p_U(\eta(s)\,\phi(s))]^{u(s)}}{r_1}\right) < \varepsilon \text{ for each } s \in S - J.$$

Let us choose r such that $d\,r_1 < r$. Then for such r, using non decreasing property of Φ, we have

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) = \Phi\left(\frac{[\,|\xi(s)|\,p_U(\phi(s))]^{u(s)}}{r}\right)$$

$$\leq \Phi\left(\frac{d\,|\eta(s)|^{u(s)}\,[p_U(\phi(s))]^{u(s)}}{r}\right)$$

$$\leq \Phi\left(\frac{[p_U(\eta(s)\,\phi(s))]^{u(s)}}{r_1}\right) < \varepsilon \text{ for all } s \in S - J.$$

Since $p_U \in D$ is an arbitrary, it clearly shows that

$\phi \in c_0(S, T, \Phi, \xi, u)$.

This proves that $c_0(S, T, \Phi, \eta, u) \subset c_0(S, T, \Phi, \xi, u)$.

Theorem 6: Let $u \in l_\infty(S, \boldsymbol{R}^+)$**.**

For any $\xi, \eta \in s(S, \mathbf{C} - \{0\})$ such that

$c_0(S, T, \Phi, \eta, u) \subset c_0(S, T, \Phi, \xi, u)$, then

$\limsup_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} < \infty$.

Proof:

Assume that $c_0(S, T, \Phi, \eta, u) \subset c_0(S, T, \Phi, \xi, u)$

but $\limsup_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} = \infty$.

Then there exists a sequence $< s_k >$ of distinct points in S such that for each $k \geq 1$,

$$|\xi(s_k)|^{u(s_k)} > k\,|\eta(s_k)|^{u(s_k)} \quad \ldots\ldots\ldots(3)$$

Now, we choose $t \in T$ and $p_V \in D$ with $p_V(t) = 1$ and define $\phi : S \to T$ by

$$\phi(s) = \begin{cases} (\eta(s))^{-1}\,k^{-1/u(s)}\,t, & \text{for } s = s_k,\ k = 1, 2, \ldots, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \quad \ldots(4)$$

Let $r > 0$. Then for each $p_U \in D$ and each $k \geq 1$, we have

$$\Phi\left(\frac{[p_U(\eta(s_k)\,\phi(s_k))]^{u(s_k)}}{r}\right) = \Phi\left(\frac{[p_U(k^{-1/u(s_k)}\,t)]^{u(s_k)}}{r}\right)$$

$$= \Phi\left(\frac{1}{k\,r}[p_U(t)]^{u(s_k)}\right) \leq \frac{1}{k}\Phi\left(\frac{[p_U(t)]^{u(s_k)}}{r}\right)$$

$$\leq \frac{1}{k}\Phi\left(\frac{A\,[(p_U(t))^{L(u)}]}{r}\right) \to 0, \text{ as } k \to \infty \text{ and}$$

$$\Phi\left(\frac{[p_U(\eta(s)\,\phi(s))]^{u(s)}}{r}\right) = 0, \text{ for } s \neq s_k,\ k \geq 1.$$

Thus for given $\varepsilon > 0$, we can find a finite subset J of S such that

$$\Phi\left(\frac{[p_U(\eta(s)\,\phi(s))]^{u(s)}}{r}\right) < \varepsilon \text{ for all } s \in S - J.$$

This shows that $\phi \in c_0(S, T, \Phi, \eta, u)$. But on the other hand for each $k \geq 1$ and in view of equations (3) and (4), we have

$$\Phi\left(\frac{[p_V(\xi(s_k)\,\phi(s_k))]^{u(s_k)}}{r}\right)$$

$$= \Phi\left(\frac{[p_V(\xi(s_k)\,(\eta(s_k))^{-1}\,k^{-1/u(s_k)}\,t)]^{u(s_k)}}{r}\right)$$

$$= \Phi\left(\frac{1}{k\,r}\left|\frac{\xi(s_k)}{\eta(s_k)}\right|^{u(s_k)}[p_V(t)]^{u(s_k)}\right) \geq \Phi\left(\frac{1}{r}\right),$$

which is independent of k.

This shows that $\phi \notin c_0(S, T, \Phi, \xi, u)$, a contradiction.

This completes the proof.

On combining the Theorems 5 and 6, we get:

Theorem 7: Let $u \in l_\infty(S, \boldsymbol{R}^+)$. Then for any $\xi, \eta \in s(S, \boldsymbol{C} - \{0\})$, $c_0(S, T, \Phi, \eta, u) \subset c_0(S, T, \Phi, \xi, u)$ if and only if

$\limsup_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} < \infty$.

When Theorems 4 and 7 are combined, we get:

Theorem 8: Let $u \in l_\infty(S, \boldsymbol{R}^+)$. Then for any $\xi, \eta \in s(S, \boldsymbol{C} - \{0\})$, $c_0(S, T, \Phi, \xi, u) = c_0(S, T, \Phi, \eta, u)$ if and only if

$$0 < \liminf_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} \leq \limsup_s \left|\frac{\xi(s)}{\eta(s)}\right|^{u(s)} < \infty.$$

Corollary 9: For $u \in l_\infty(S, \boldsymbol{R}^+)$ and $\xi \in s(S, \boldsymbol{C} - \{0\})$. Then

(i) $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, u)$ if and only if $\liminf_s |\xi(s)|^{u(s)} > 0$;

(ii) $c_0(S, T, \Phi, u) \subset c_0(S, T, \Phi, \xi, u)$ if and only if $\limsup_s |\xi(s)|^{u(s)} < \infty$; and

(iii) $c_0(S, T, \Phi, \xi, u) = c_0(S, T, \Phi, u)$ if and only if $0 < \liminf_s |\xi(s)|^{u(s)} \le \limsup_s |\xi(s)|^{u(s)} < \infty$.

Proof:

If we consider, $\eta : S \to C - \{0\}$ such that $\eta(s)=1$ for each s, in Theorems 4, 7 and 8, we easily obtain the assertions (i), (ii) and (iii) respectively.

Theorem 10: If $u \in l_\infty(S, \boldsymbol{R}^+)$, $w : S \to \boldsymbol{R}^+$ and $\xi \in s(S, \boldsymbol{C} - \{0\})$, then

$c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \xi, w)$ if $\liminf_s \frac{w(s)}{u(s)} > 0$.

Proof:

Assume that $\liminf_s \frac{w(s)}{u(s)} > 0$. Then there exists $m > 0$ such that $w(s) > m\, u(s)$ for all but finitely many $s \in S$.

Let $\phi \in c_0(S, T, \Phi, \xi, u)$, $r > 0$ be associated with ϕ and $\varepsilon > 0$.

Then for $0 < \rho < 1$ with $\rho^m \Phi\left(\frac{1}{r}\right) < \varepsilon$ and $p_U \in D$, there exists $J \in \mathcal{F}(S)$ satisfying

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) < \Phi\left(\frac{\rho}{r}\right) \text{ for each } s \in S - J.$$

Since Φ is non decreasing, therefore,

$[p_U(\xi(s)\,\phi(s))]^{u(s)} < \rho$ and so

$[p_U(\xi(s)\,\phi(s))]^{w(s)} \le [(p_U(\xi(s)\,\phi(s)))^{u(s)}]^m < \rho^m$

Hence using convexity of Φ, we have

$$\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{w(s)}}{r}\right) \le \Phi\left(\frac{\rho^m}{r}\right)$$

$$\le \rho^m\, \Phi\left(\frac{1}{r}\right) < \varepsilon, \text{ for each } s \in S - J.$$

Since $p_U \in D$ is arbitrary, we easily get:

$\phi \in c_0(S, T, \Phi, \xi, w)$.

Hence, $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \xi, w)$.

Theorem 11: If $u \in l_\infty(S, \boldsymbol{R}^+)$, $w : S \to \boldsymbol{R}^+$, $\xi \in s(S, \boldsymbol{C} - \{0\})$ such that $c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \xi, w)$, then

$\liminf_s \frac{w(s)}{u(s)} > 0$.

Proof:

Assume that the inclusion,

$c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \xi, w)$ holds but

$\liminf_s \frac{w(s)}{u(s)} = 0$.

Then there exists a sequence $< s_k >$ of distinct points in S such that for each $k \ge 1$,

$$k\, w(s_k) < u(s_k) \qquad \text{……. … (5)}$$

Now, taking $p_V \in D$ and $t \in T$ with $p_V(t) = 1$

define $\phi : S \to T$ by the relation

$$\phi(s) = \begin{cases} (\xi(s))^{-1} k^{-1/u(s)}\, t, & \text{for } s = s_k,\ k = 1, 2, 3, \ldots, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \qquad \text{...(6)}$$

Let $r > 0$. Then for each $p_U \in D$ and $k \ge 1$, we have

$$\Phi\left(\frac{[p_U(\xi(s_k)\,\phi(s_k))]^{u(s_k)}}{r}\right) = \Phi\left(\frac{[p_U(k^{-1/u(s_k)}\, t)]^{u(s_k)}}{r}\right)$$

$$\le \frac{1}{k}\Phi\left(\frac{[p_U(t)]^{u(s_k)}}{r}\right) \le \frac{1}{k}\Phi\left(\frac{A\,[(p_U(t))^{L(u)}]}{r}\right)$$

and for $s \ne s_k$, $k \ge 1$, $\Phi\left(\frac{[p_U(\xi(s)\,\phi(s))]^{u(s)}}{r}\right) = 0$.

This shows that $\phi \in c_0(S, T, \Phi, \xi, u)$.

On the other hand for each k ≥ 1 and in view of equations (5) and (6), we have

$$\Phi\left(\frac{[p_V(\xi(s_k)\,\phi(s_k))]^{w(s_k)}}{r}\right) = \Phi\left(\frac{[p_V(k^{-1/u(s_k)}\, t)]^{w(s_k)}}{r}\right)$$

$$\ge \Phi\left(\frac{1}{r\,k^{1/k}}\right) \ge \Phi\left(\frac{1}{r\sqrt{e}}\right).$$

This shows that $\phi \notin c_0(S, T, \Phi, \xi, w)$, a contradiction. This completes the proof.

On combining the Theorems 10 and 11, one can obtain:

Theorem 12: If $u \in l_\infty(S, \boldsymbol{R}^+)$, $w : S \to \boldsymbol{R}^+$ and $\xi \in s(S, C - \{0\})$, then

$c_0(S, T, \Phi, \xi, u) \subset c_0(S, T, \Phi, \xi, w)$ if and only if

$\liminf_s \frac{w(s)}{u(s)} > 0$.

Theorem 13: Let $u : S \to \boldsymbol{R}^+$, $w \in l_\infty(S, \boldsymbol{R}^+)$ and $\xi \in s(S, C - \{0\})$, then

$c_0(S, T, \Phi, \xi, w) \subset c_0(S, T, \Phi, \xi, u)$ if $\limsup_s \frac{w(s)}{u(s)} < \infty$.

Proof:

Assume that $\limsup_s \frac{w(s)}{u(s)} < \infty$.

Then there exists a constant $d > 0$ such that $w(s) < d\, u(s)$ for all but finitely many $s \in S$.

Let $\phi \in c_0 (S, T, \Phi, \xi, w)$, $r > 0$ be associated with ϕ and $\varepsilon > 0$.

Then for $0 < \rho < 1$ with $\rho^{1/d} \Phi\left(\frac{1}{r}\right) < \varepsilon$ and $p_U \in D$, there exists $J \in \mathcal{F}(S)$ satisfying

$$\Phi\left(\frac{[p_U(\xi(s)\phi(s))]^{w(s)}}{r}\right) < \Phi\left(\frac{\rho}{r}\right) \text{ for each } s \in S - J.$$

Since Φ is non decreasing, therefore

$[p_U(\xi(s)\phi(s))]^{w(s)} < \rho < 1$ and so

$[p_U(\xi(s)\phi(s))]^{u(s)} \le [(p_U(\xi(s)\phi(s)))^{w(s)}]^{1/d} < \rho^{1/d}$.

Hence using the convexity of Φ, we have

$$\Phi\left(\frac{[p_U(\xi(s)\phi(s))]^{u(s)}}{r}\right) \le \Phi\left(\frac{\rho^{1/d}}{r}\right)$$

$$\le \eta^{1/d} \Phi\left(\frac{1}{r}\right) < \varepsilon \text{ for each } s \in S - J.$$

Since $p_U \in D$ is arbitrary, this clearly implies that $\phi \in c_0 (S, T, \Phi, \xi, u)$ and hence

$c_0(S, T, \Phi, \xi, w) \subset c_0 (S, T, \Phi, \xi, u)$.

This completes the proof.

Theorem 14: Let $u : S \to \boldsymbol{R}^+$, $w \in l_\infty (S, R^+)$, $\xi \in s (S, C - \{0\})$ and $c_0 (S, T, \Phi, \xi, w) \subset c_0 (S, T, \Phi, \xi, u)$, then $\limsup_s \frac{w(s)}{u(s)} < \infty$.

Proof:

Suppose that $c_0 (S, T, \Phi, \xi, w) \subset c_0(S, T, \Phi, \xi, u)$ but $\limsup_s \frac{w(s)}{u(s)} = \infty$.

Then there exists a sequence $< s_k >$ in S of distinct points such that

$w(s_k) > k\, u(s_k)$ for each $k \ge 1$. . ………… (7)

Now, taking $p_V \in D$ and $t \in T$ with $p_V(t) = 1$.

We define $\phi : S \to T$ by

$$\phi(s) = \begin{cases} (\xi(s))^{-1} k^{-1/w(s)}\, t, & \text{for } s = s_k, k \ge 1 \\ \theta, & \text{otherwise.} \end{cases} \quad \text{……..(8}$$

Then analogous to the proof of Theorem 11 and in view of equations (7) and (8), we can show that

$\phi \in c_0 (S, T, \Phi, \xi, w)$ and $\phi \notin c_0 (S, T, \Phi, \xi, u)$, a contradiction.

The proof is now complete.

On combining the Theorems 13 and 14, one can obtain:

Theorem 15: Let $u : S \to \boldsymbol{R}^+$, $w \in l_\infty (S, \boldsymbol{R}^+)$ and $\xi \in s (S, \boldsymbol{C} - \{0\})$, then $c_0 (S, T, \Phi, \xi, w) \subset c_0 (S, T, \Phi, \xi, u)$ if and only if $\limsup_s \frac{w(s)}{u(s)} < \infty$.

CONCLUSION

Present paper examined some conditions that characterize the linear space structures and containment relations on the locally convex topological vector space valued null functions defined by semi norm and Orlicz function. In fact, these results can be used for further generalization to investigate other properties of the function spaces using Orlicz function.

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