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Research Article

SUB-GAUSSIAN TYPE ESTIMATES FOR SYMMETRIC RANDOM VARIABLE

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ABSTRACT

Sub-Gaussian type estimates are crucial when investigating the asymptotic behavior of symmetric independent random variables. This article aims to establish some sub-Gaussian type estimates for these variables, specifically for the summation of first n number of variables and for the tail sums of those variables. We derive three such estimates.

Keywords: Gaussian-type estimate, law of the iterated logarithm, Levy's inequality, symmetric random variables

INTRODUCTION

In various real-world situations, the distribution of a random variable may deviate significantly from a Gaussian distribution. In such cases, sub-Gaussian type estimates can be used to characterize the behavior of the random variables. In the theory of probability and statistics, a sub-Gaussian type estimate is a mathematical property which characterizes the behavior of independent random variables. It provides a measure of how much the given variables deviate from their mean along with how closely their distribution is concentrated around the mean. More precisely, a sub-Gaussian type estimate implies that the distribution of the random variable decays in exponential order. Consequently, the variable has limited variance and can not deviate significantly from its mean. This type of property is considered to be very useful whenever we need to accurately estimate the property of the random variables. The sub-Gaussian estimate has numerous applications in areas such as statistical inference, signal processing, machine learning and the law of the iterated logarithm. Additionally, sub-Gaussian estimates are necessary in the examination of high-dimensional data, where the number of variables is much greater than the number of observations. In this context, the law of the iterated logarithm can be used to control the deviation of the empirical mean from the true mean where we can use the sub-Gaussian type estimates to justify that the law of the iterated logarithm holds with high probability. Many important classes of random variables satisfy sub-Gaussian type estimates including Gaussian random variables and bounded random variables.

In this article, we derive three sub-Gaussian type estimates for symmetric, bounded, and independent random variables. These types of estimates are useful while deriving the law of the iterated logarithm for the sums of independent random variables (Chung, 2005). One can find various applications of law of the iterated logarithm in various contexts. Please see Mingzhou and Cheng (2022), Chen and Qi (2006) and Schatte (1988) for the details. Before we prove our main results, we first recall some definitions and theorems.

Definition 1.1 [Symmetric random variable] A random variable is a function that assigns a number to each point in a sample space. Thus, a random variable is a function from a sample space to the set of real numbers and is usually denoted by X. A random variable X is said to be bounded if there exists a real number $M > 0$ such that for all $s \in S$ (sample space), we have $|X(s)| \leq M$. The random variable X is said to be symmetric if and only if both the functions X and $-X$ have the same distribution (Chung, 2005).

Definition 1.2 [Independent random variable] Let (X, Y) be a bivariate random vector with joint probability density function $f(x, y)$ and marginal probability density function $f_X(x)$ and $f_Y(y)$. Then the random variables X and Y are said to be independent random variables if for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we have $f(x, y) = f_X(x). f_Y(y)$ (Casella & Berger, 2002).

Definition 1.3 [Martingales] A sequence of random variables and Borel Fields $\{X_n, \mathfrak{F}_n\}$ is said to a martingale if and only if the following conditions are satisfied:

i. $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$ for all n and $X_n \in \mathfrak{F}_n$;
ii. $E(|X_n|) < \infty$;
iii. $E(X_{n+1}|\mathfrak{F}_n) = X_n$ a.e.

where a.e. stands for almost everywhere equal. The sequence is called the submartingale if and only if the equality $=$ in (iii) is replaced by $\leq$ and the sequence is called the super martingale if and only if the equality $=$ is replaced by $\geq$ (Banelos & Moore, 1991).

Definition 1.4 [Law of the Iterated Logarithm] The Law of the Iterated Logarithm (LIL) is a mathematical theorem that describes the behavior of the partial sums of independent and identically distributed random variables. It is a stronger version of the law of large numbers and provides information about the fluctuations of the sum of random variables around its expected value. For a sequence of independent and identically distributed random variables $X_1, X_2, \ldots . X_n$ with mean zero and finite variance σ^2, the law of the iterated logarithm states that as the number of terms in the sum increases, the normalized sum oscillates between $-\sqrt{2\,\sigma^2 \log\log n}$ and $\sqrt{2\,\sigma^2 \log\log n}$ infinitely often, almost surely. Mathematically, it can be expressed as:

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2\,\sigma^2 \log\log n}} = 1$$

a.s.

Theorem 1.5 [Continuity Property of measure] *If $\{A_n\}$ is a sequence of sets on a sigma algebra with the property that $A_n \subset A_{n+1}$ for all n and $A = \bigcup_{n=1}^{\infty} A_n$, then we have*

$|A| = \lim_{n\to\infty} |A_n|$ (Royden & Fitzpatrick, 2010).

Lemma 1.6 *If Y_i are independent random variable with mean $E(Y_i) = 0$, then the sum of Y_i i.e. $S_m = \sum_{i=1}^{m} Y_i$ is a martingale and S_m^2 is a submartingale* (Chung, 2005).

Lemma 1.7 *Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of submartingale and φ be a convex function which is increasing on $\mathbb{R}$. If φ is integrable function on $\mathbb{R}$, then the function $\varphi(Y_i)$ is a submartingale* (Chung, 2005).

Theorem 1.8 [Doob's Maximal Identity] *If $\{Y_i\}_{i=1}^{\infty}$ is a sequence of submartingale, then for any $N > 0$, we have*

$$\left|\left\{\omega: \max_{1\le i\le M} Y_i \ge N\right\}\right| \le \frac{1}{N}\, E(max(Y_i, 0)).$$

Theorem 1.9 [Hoeffding] *Let $\{Y_i\}_{i=1}^{m}$ be independent random variables with mean zero and bounded ranges such that $a_i \le Y_i \le b_i$ for all $i = 1, 2, ..m$. Then for each $\lambda > 0$, we have*

$$|\{\omega: |\textstyle\sum_{i=1}^{m} Y_i| > \lambda\}| \le 2\, exp\left(\frac{-2\,\lambda^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right).$$

Theorem 1.10 [Levy's Inequality] *If $X_1, X_2 \ldots X_n$ be independent and symmetric random variables. Let $S_m = X_1 + X_2 + \cdots + X_m$. Then for all $\lambda > 0$, we have*

$$P\left(\max_{1\le k\le m} |S_k| \ge \lambda\right) \le 2\, P(|S_m| \ge \lambda)$$
$$P\left(\max_{1\le k\le m} |X_k| \ge \lambda\right) \le 2\, P(|S_m| \ge \lambda).$$

MAIN RESULTS

In this section, we derive three estimates for the sums of independent, bounded, symmetric and identically distributed random variables with mean zero and variance one. We begin with the first estimate.

Theorem 2.1 *Let $\{Y_i\}_{i=1}^{\infty}$ be random variables which are independent, bounded, symmetric and identically distributed with mean zero and variance 1 such that $-1 \le Y_i \le$, $X_i = a_i Y_i$ and $\{a_i\}_{i=1}^{\infty}$ are real constants, then for all $\alpha > 0, \beta > 0$, we have*

$$\left|\left\{t: \sup_{1\le n\le m} \left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \le A.\, N(\alpha)\, exp\left(\frac{(-1 + 2\alpha)\beta^2}{2\,\sum_{i=1}^{m} a_i^2}\right).$$

for some constant A and number $N(\alpha)$, depending on α.

Proof: Using an elementary fact " $\sup|A| > \beta$ implies that $\sup A > \beta$ or $\sup(-A) > \beta$", we have

$$\left\{t: \sup_{1\le n\le m} \left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\} = \left\{t: \sup_{1\le n\le m} \sum_{k=1}^{n} X_k(t) > \beta\right\} \cup \left\{t: \sup_{1\le n\le m} -\sum_{k=1}^{n} X_k(t) > \beta\right\}.$$

So, for any $\eta > 0$, we have

$$\left|\left\{t: \sup_{1\le n\le m} \left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \le \left|\left\{t: \sup_{1\le n\le m} \sum_{k=1}^{n} X_k(t) > \beta\right\}\right| + \left|\left\{t: \sup_{1\le n\le m} -\sum_{k=1}^{n} X_k(t) > \beta\right\}\right|$$
$$\le \left|\left\{t: \sup_{1\le n\le m} \exp\left(\eta \sum_{k=1}^{n} X_k(t)\right) > e^{\eta\beta}\right\}\right| + \left|\left\{t: \sup_{1\le n\le m} \exp\left(-\eta \sum_{k=1}^{n} X_k(t)\right) > e^{\eta\beta}\right\}\right|$$

Employing Lemma 1.5, the sum $\sum_{k=1}^{n} X_k$ is a martingale. Moreover, the function $e^{\eta x}$ is convex and an increasing function. Then the functions $\exp(\eta \sum_{k=1}^{n} X_k(t))$ and $\exp(-\eta \sum_{k=1}^{n} X_k(t))$ are both submartingales. Let μ denote Lebesgue's measure. Then using Doob's maximal inequality (Theorem 1.7) for submartingales, we get

$$\left|\left\{t:\sup_{1\le n\le m}\left|\sum_{k=1}^{n}X_k(t)\right|>\beta\right\}\right|\le\frac{1}{e^{\eta\beta}}\int_I\exp\left(\eta\sum_{k=1}^{m}X_k(t)\right)d\mu+\frac{1}{e^{\eta\beta}}\int_I\exp\left(\eta\sum_{k=1}^{m}X_k(t)\right)d\mu$$
$$\le\frac{2}{e^{\eta\beta}}\int_I\exp\left(\eta\sum_{k=1}^{m}X_k(t)\right)d\mu$$

Hence, we have

$$\left|\left\{t:\sup_{1\le n\le m}\left|\sum_{k=1}^{n}X_k(t)\right|>\beta\right\}\right|\le\frac{2}{e^{\eta\beta}}\int_I\exp\left(\eta\sum_{k=1}^{m}X_k(t)\right)d\mu\ \ldots\ldots\ldots\ldots.(1)$$

Employing Hoeffding's theorem (Theorem 1.8), we have

$$\left|\left\{t:\left|\sum_{k=1}^{m}X_k(t)\right|\ge\beta\right\}\right|\le 2\exp\left(-\frac{\beta^2}{2\sum_{i=1}^{m}a_i^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Now for all $\alpha>0$ and for $I=[0,1)$, we claim that:

$$\int_I\exp\left(\eta\sum_{k=1}^{m}X_k(t)\right)d\mu\le M\sqrt{8\pi}\exp\left(\left[\frac{1}{2}+\alpha\right]\eta^2\sum_{i=1}^{m}a_i^2\right)$$

With the help of Fubini's theorem, one can show that

$$\int_I e^g\,d\mu=\int_{-\infty}^{\infty}e^{\beta}\,\mu(\{t:g(t)>\beta\})\,d\beta\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

Then from (2) and (3), we get

$$\int_I\exp\left(\eta\left|\sum_{k=1}^{m}X_k(t)\right|\right)d\mu$$
$$=\int_{-\infty}^{\infty}e^{\beta}\,\mu\left(\left\{t:\eta\left|\sum_{k=1}^{m}X_k(t)\right|>\beta\right\}\right)d\beta$$
$$=\int_{-\infty}^{\infty}e^{\beta}\,\mu\left(\left\{t:\left|\sum_{k=1}^{m}X_k(t)\right|>\frac{\beta}{\eta}\right\}\right)d\beta$$
$$\le\int_{-\infty}^{\infty}e^{\beta}\,\exp\left(-\frac{\beta^2}{2\eta^2\sum_{i=1}^{m}a_i^2}\right)d\beta$$
$$=2\int_{-\infty}^{\infty}\exp\left(\frac{-1}{2\eta^2\sum_{i=1}^{m}a_i^2}\left[\beta^2-2\eta^2\sum_{i=1}^{m}a_i^2\,\beta\right]\right)d\beta$$
$$=2\int_{-\infty}^{\infty}\exp\left(\frac{-1}{2\eta^2\sum_{i=1}^{m}a_i^2}\left[\beta^2-2\eta^2\sum_{i=1}^{m}a_i^2\,\beta\right]\pm\left(\eta^2\sum_{i=1}^{m}a_i^2\right)^2\right)d\beta$$
$$=2\exp\left(\frac{\eta^2\sum_{i=1}^{m}a_i^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(\frac{-[\beta-\eta^2\sum_{i=1}^{m}a_i^2]^2}{2\,\eta^2\sum_{i=1}^{m}a_i^2}\right)d\beta$$

Setting $v=\frac{\beta-\eta^2\sum_{i=1}^{m}a_i^2}{\eta\sqrt{\sum_{i=1}^{m}a_i^2}}$, we get $\eta\sqrt{\sum_{i=1}^{m}a_i^2}\,dv=d\beta$. Then the above relation becomes:

$$\int_I\exp\left(\eta\left|\sum_{k=1}^{m}X_k(t)\right|\right)d\mu=2\exp\left(\frac{\eta^2\sum_{i=1}^{m}a_i^2}{2}\right)\int_{-\infty}^{\infty}e^{\frac{-v^2}{2}}\,\eta\sqrt{\sum_{i=1}^{m}a_i^2}\,dv$$
$$=2\eta\sqrt{\sum_{i=1}^{m}a_i^2}\exp\left(\frac{\eta^2\sum_{i=1}^{m}a_i^2}{2}\right)\int_{-\infty}^{\infty}e^{\frac{-v^2}{2}}\,dv$$
$$=2\sqrt{2\pi}\,\eta\sqrt{\sum_{i=1}^{m}a_i^2}\exp\left(\frac{\eta^2\sum_{i=1}^{m}a_i^2}{2}\right)$$

Thus, we have

$$\int_I \exp\left(\eta\left|\sum_{k=1}^{m} X_k(t)\right|\right) d\mu \le 2\sqrt{2\pi}\,\eta\sqrt{\sum_{i=1}^{m} a_i^2}\exp\left(\frac{\eta^2\sum_{i=1}^{m} a_i^2}{2}\right).$$

We recall that for a given $\alpha > 0$, there exists $N = N(\alpha) > 0$ such that for all $V > 0$,

$$V.\exp\left(\frac{V^2}{2}\right) \le N(\alpha)\exp\left(\left[\frac{1}{2}+\alpha\right]V^2\right)$$

Employing this inequality in the above relation, we have

$$\int_I \exp\left(\eta\left|\sum_{k=1}^{m} X_k(t)\right|\right) d\mu \le 2\sqrt{2\pi}\;N(\alpha)\exp\left(\left[\frac{1}{2}+\alpha\right]\eta^2\sum_{i=1}^{m} a_i^2\right) \dots\dots\dots\dots\dots\dots\dots (4)$$

Let us choose $\eta = \frac{\beta}{\sum_{i=1}^{m} a_i^2}$. Using this η in equation (1) and (4), we have

$$\begin{aligned}\left|\left\{t\colon \sup_{1\le n\le m}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| &\le \frac{2}{e^{\eta\beta}}\int_I \exp\left(\eta\sum_{k=1}^{m} X_k(t)\right) d\mu \\ &\le \frac{2}{e^{\eta\beta}}\;2\sqrt{2\pi}\;N(\alpha)\exp\left(\left[\frac{1}{2}+\alpha\right]\eta^2\sum_{i=1}^{m} a_i^2\right) \\ &= \sqrt{8\pi}\;N(\alpha)\exp\left(\left[\frac{1}{2}+\alpha-1\right]\;\frac{\beta^2}{\sum_{i=1}^{m} a_i^2}\right) \\ &= \sqrt{8\pi}\;N(\alpha)\exp\left(\left[\frac{-1}{2}+\alpha\right]\;\frac{\beta^2}{\sum_{i=1}^{m} a_i^2}\right).\end{aligned}$$

Hence, we have

$$\left|\left\{t\colon \sup_{1\le n\le m}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \le A.N(\alpha)\exp\left(\frac{(-1+2\alpha)\beta^2}{2\sum_{i=1}^{m} a_i^2}\right).$$

With the above estimate in hand, we now use continuity property of Lebesgue measure to derive the following estimate:

Theorem 2.2 *Let* $\{Y_i\}_{i=1}^{\infty}$ *be random variables which are independent, bounded, symmetric and identically distributed with mean zero and variance 1 such that* $-1 \le Y_i \le,\ X_i = a_i Y_i$ *and* $\{a_i\}_{i=1}^{\infty}$ *are real constants, then for all* $\alpha > 0, \beta > 0,$ *we have*

$$\left|\left\{t\colon sup_{n\ge 1}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \le A\,N(\alpha).exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=1}^{\infty} a_i^2}\right)$$

for some constant for some constant A *and number* $N(\alpha)$*, depending on* α*.*
Proof: Using Theorem 2.1, we have

$$\left|\left\{t\colon \sup_{1\le n\le m}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \le A.N(\alpha)\exp\left(\frac{(-1+2\alpha)\beta^2}{2\sum_{i=1}^{m} a_i^2}\right)$$

Let $A_m = \left\{t\colon \sup_{1\le n\le m}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}$ and let us write $A = \bigcup_{m=1}^{\infty} A_m$. One can check that $A_m \subset A_{m+1}$ which follows simply by the property of supremum. By continuity property of Lebesgue measure (Theorem 1.4), we have $|A| = \lim_{m\to\infty} |A_m|$. This gives

$$\begin{aligned}\left|\left\{t\colon \sup_{n\ge 1}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| &= \lim_{m\to\infty} |A_m| \\ &= \lim_{m\to\infty}\left|\left\{t\colon \sup_{1\le n\le m}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right|\end{aligned}$$

$$\leq \lim_{m\to\infty} A.N(\alpha)\exp\left(\frac{(-1+2\alpha)\beta^2}{2\sum_{i=1}^{m} a_i^2}\right)$$
$$\leq A.N(\alpha)\exp\left(\frac{(-1+2\alpha)\beta^2}{2\sum_{i=1}^{\infty} a_i^2}\right)$$

Thus, we have

$$\left|\left\{t: \sup_{n\geq 1}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \leq A.N(\alpha)\exp\left(\frac{(-1+2\alpha)\beta^2}{2\sum_{i=1}^{\infty} a_i^2}\right)$$

With the appropriate choice of α, we have

$$\left|\left\{t: \sup_{n\geq 1}\left|\sum_{k=1}^{n} X_k(t)\right| > \beta\right\}\right| \leq A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=1}^{\infty} a_i^2}\right).$$

With the help of the above estimate, we now derive an estimate for tail sums of independent random variables as follows.

Theorem 2.3 *Let* $\{Y_i\}_{i=1}^{\infty}$ *be random variables which are independent, bounded, symmetric and identically distributed with mean zero and variance 1 such that* $-1 \leq Y_i \leq$, $X_i = a_i Y_i$ *and* $\{a_i\}_{i=1}^{\infty}$ *are real constants, then for all* $\alpha > 0, \beta > 0$, *we have*

$$\left|\left\{t: sup_{n\geq m}\left|\sum_{i=n+1}^{\infty} X_i(t)\right| > \beta\right\}\right| \leq A\,N(\alpha).\,exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right)$$

for some constant A *and number* $N(\alpha)$, *depending on* α.

Proof: Let m be fixed. Then define b_i as follows:

$$b_i = \begin{cases} 0\,, & if\ i \leq m \\ a_i, & if\ i > m \end{cases}$$

Using Theorem 2.2, for the sequence $\{\sum_{i=1}^{n} b_i Y_i\}$ we have

$$\left|\left\{t: \sup_{n\geq 1}\left|\sum_{i=1}^{n} b_i Y_i(t)\right| > \beta\right\}\right| \leq A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=1}^{\infty} b_i^2}\right)$$

As $b_i = 0$ for $i \geq m$, we get

$$\left|\left\{t: \sup_{n\geq m}\left|\sum_{i=m+1}^{n} a_i Y_i(t)\right| > \beta\right\}\right| \leq A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right)$$

This gives

$$\left|\left\{t: \sup_{n\geq m}\left|\sum_{i=1}^{n} a_i Y_i(t) - \sum_{i=1}^{m} a_i Y_i(t)\right| > \beta\right\}\right| \leq A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right)$$
$$\left|\left\{t: \sup_{n\geq m}\left|\sum_{i=1}^{n} X_i(t) - \sum_{i=1}^{m} X_i(t)\right| > \beta\right\}\right| \leq A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\sum_{i=m+1}^{\infty} a_i^2}\right) \ldots\ldots\ldots\ldots\ldots(5)$$

Let $M \gg m$ where m is fixed number. Applying Levy's inequality (Theorem 1.9), we get

$$\left|\left\{t: \sup_{m\leq j\leq M-1}\left|\sum_{i=1}^{j-m} X_{M-i}(t)\right| > \beta\right\}\right| \leq 2\ \left|\left\{t: \left|\sum_{i=0}^{M-m-1} X_{M-i}(t)\right| > \beta\right\}\right|$$
$$\left|\left\{t: \sup_{m\leq n\leq M-1}\left|\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta\right\}\right| \leq 2\ \left|\left\{t: \left|\sum_{i=0}^{M} X_{M-i}(t) - \sum_{i=1}^{m} X_i(t)\right| > \beta\right\}\right|$$

Thus,

$$\left|\left\{t: \sup_{m\leq n\leq M}\left|\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta\right\}\right| \leq 2\ \left|\left\{t: \left|\sum_{i=0}^{M} X_{M-i}(t) - \sum_{i=1}^{m} X_i(t)\right| > \beta\right\}\right| \ldots\ldots(6)$$

Since $M \gg m,$ we have (using (5))

$$\left|\left\{t:\ \left|\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{m} X_i(t)\right| > \beta\right\}\right| \leq\ A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\ \sum_{i=m+1}^{\infty} a_i^2}\right) \ldots\ldots\ldots\ldots. (7)$$

From the equations (6) and (7), we have

$$\left|\left\{t:\ \sup_{m\leq n\leq M}\left|\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta\right\}\right| \leq\ 2\ A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\ \sum_{i=m+1}^{\infty} a_i^2}\right)$$

Let us define $A_M = \left\{t:\ \sup_{m\leq n\leq M} |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)| > \beta\right\}$ and $A = \cup_{M=1}^{\infty} A_M$.

Then using Theorem 1.4, we get $\lim_{M\to\infty} |A_M| = |A|$.

If we choose M sufficiently large, then for this $M,$ we have

$$\sup_{n\geq m}\left|\sum_{i=1}^{\infty} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta\ \Rightarrow\ \sup_{M\geq n\geq m}\left|\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta$$

Consequently, we have $t \in A_M$ for sufficiently large $M.$ This means that $t \in A.$ Hence

$$\begin{aligned}\left|\left\{t:\ \sup_{n\geq m}\left|\sum_{i=1}^{\infty} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta\right\}\right| &\leq |A| \\ &\leq \lim_{M\to\infty} |A_M| \\ &\leq \lim_{M\to\infty}\left|\left\{t:\ \sup_{M\geq n\geq m} |\sum_{i=1}^{M} X_i(t) - \sum_{i=1}^{n} X_i(t)| > \beta\right\}\right| \\ &\leq \lim_{M\to\infty} 2\ A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\ \sum_{i=m+1}^{\infty} a_i^2}\right) \\ &= 2\ A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\ \sum_{i=m+1}^{\infty} a_i^2}\right)\end{aligned}$$

This gives,

$$\left|\left\{t:\ \sup_{n\geq m}\left|\sum_{i=1}^{\infty} X_i(t) - \sum_{i=1}^{n} X_i(t)\right| > \beta\right\}\right| \leq 2\ A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\ \sum_{i=m+1}^{\infty} a_i^2}\right).$$

Thus, we have

$$\left|\left\{t:\ \sup_{n\geq m}\left|\sum_{i=n+1}^{\infty} X_i(t)\right| > \beta\right\}\right| \leq 2\ A.N(\alpha)\exp\left(\frac{(-1+\alpha)\beta^2}{2\ \sum_{i=m+1}^{\infty} a_i^2}\right).$$

This completes the proof of the Theorem.

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CONFLICT OF INTEREST

The authors do not have any conflict of interest pertinent to this work.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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