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Inverse of an Arbitrary Vandermonde Matrix

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Abstract. It is shown a general formula to invert any type Vandermonde matrix.

The Vandermonde [1-3] matrices have the structure:

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}, \cdots \tag{1}$$

and are useful in the Lagrange interpolation [4]. Here it is exhibited a general expression to construct the inverse of a matrix of the type (1); in fact, let

$$\underset{\sim}{V} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}, \quad n = 2, 3, \cdots \tag{2}$$

from which it is well known [2] that its determinant is not zero if the x_j are different among them, then under this last assumption there exists $\underset{\sim}{V}^{-1} = \left(A_{rk} \right)$, and for their elements it is presented the following formula

$$A_{rk} = \frac{(-1)^{n+r}}{B_k} C_{rk}\,, \tag{3}$$

where:

$$B_k = \prod_{\substack{j=1 \\ j\neq k}}^{n} \left(x_k - x_j \right), \quad C_{nk} = 1, \quad k = 1, 2, \cdots, n$$

$$C_{rk} = \sum_{i_1,\cdots,i_c} x_{i_1} x_{i_2} \cdots x_{i_c}, \quad \begin{matrix} c = n - r, & r = 1, \cdots, n-1, \\ i_j \neq k, & j = 1, \cdots, c, \end{matrix} \quad , \tag{4}$$

$$i_1 = n, n-1, \cdots, c, \quad i_2 = i_1 - 1, \cdots, 1 \cdots, \quad i_c = i_{c-1} - 1, \cdots, 1,$$

which for n = 2,3,4,… leads to the inverse matrices:

$$\frac{1}{B_2}\begin{pmatrix} x_2 & -x_1 \\ -1 & 1 \end{pmatrix}, \quad B_2 = -B_1 = x_2 - x_1,$$

$$\begin{pmatrix} \frac{x_2 x_3}{B_1} & \frac{x_1 x_3}{B_2} & \frac{x_1 x_2}{B_3} \\ -\frac{x_2 + x_3}{B_1} & -\frac{x_1 + x_3}{B_2} & -\frac{x_1 + x_2}{B_3} \\ \frac{1}{B_1} & \frac{1}{B_2} & \frac{1}{B_3} \end{pmatrix}, \quad \begin{matrix} B_1 = \left(x_1 - x_2 \right)\left(x_1 - x_3 \right), \\ B_2 = \left(x_2 - x_1 \right)\left(x_2 - x_3 \right), \\ B_3 = \left(x_3 - x_2 \right)\left(x_3 - x_1 \right), \end{matrix} \tag{5}$$

$$\left(\begin{matrix} -\frac{x_2 x_3 x_4}{B_1} & -\frac{x_1 x_3 x_4}{B_2} & \frac{x_1 x_2 x_4}{B_3} & \frac{x_1 x_2 x_3}{B_4} \\ \frac{x_2 x_3 + x_2 x_4 + x_3 x_4}{B_1} & \frac{x_1 x_3 + x_1 x_4 + x_3 x_4}{B_2} & \frac{x_1 x_2 + x_1 x_4 + x_2 x_4}{B_3} & \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{B_4} \\ -\frac{x_2 + x_3 + x_4}{B_1} & -\frac{x_1 + x_3 + x_4}{B_2} & \frac{x_1 + x_2 + x_4}{B_3} & \frac{x_1 + x_2 + x_3}{B_4} \\ \frac{1}{B_1} & \frac{1}{B_2} & \frac{1}{B_3} & \frac{1}{B_4} \end{matrix} \right.$$

with:

$$B_1 = \left(x_1 - x_2 \right)\left(x_1 - x_3 \right)\left(x_1 - x_4 \right), \quad B_2 = \left(x_2 - x_1 \right)\left(x_2 - x_3 \right)\left(x_2 - x_4 \right),$$
$$B_3 = \left(x_3 - x_1 \right)\left(x_3 - x_2 \right)\left(x_3 - x_4 \right), \quad B_4 = \left(x_4 - x_1 \right)\left(x_4 - x_2 \right)\left(x_4 - x_3 \right),$$

which in turn give the identity matrix after multiplication by (1).

Turner [5], see its relations (6) and (8), showed that $V_{n\times n}^{-1}$ is the product of two triangular matrices, which can be illustrated with the inverse matrices (5):

$$V_{2\times 2}^{-1} = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{B_1} & \frac{1}{B_2} \end{pmatrix},$$

$$V_{3\times 3}^{-1} = \begin{pmatrix} 1 & -x_1 & x_1 x_2 \\ 0 & 1 & -\left(x_1 + x_2\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{1}{x_1 - x_2} & \dfrac{1}{x_2 - x_1} & 0 \\ \dfrac{1}{B_1} & \dfrac{1}{B_2} & \dfrac{1}{B_3} \end{pmatrix}, \qquad (6)$$

$$V_{4\times 4}^{-1} = \begin{pmatrix} 1 & -x_1 & x_1 x_2 & -x_1 x_2 x_3 \\ 0 & 1 & -\left(x_1 + x_2\right) & x_1 x_2 + x_2 x_3 + x_1 x_3 \\ 0 & 0 & 1 & -\left(x_1 + x_2 + x_3\right) \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \underset{\sim}{T},$$

with:

$$\underset{\sim}{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \dfrac{1}{x_1 - x_2} & \dfrac{1}{x_2 - x_1} & 0 & 0 \\ \dfrac{1}{\left(x_1 - x_2\right)\left(x_1 - x_3\right)} & \dfrac{1}{\left(x_2 - x_1\right)\left(x_2 - x_3\right)} & \dfrac{1}{\left(x_3 - x_1\right)\left(x_3 - x_2\right)} & 0 \\ \dfrac{1}{B_1} & \dfrac{1}{B_2} & \dfrac{1}{B_3} & \dfrac{1}{B_4} \end{pmatrix}.$$

The expressions (3) and (4) are easy to manage with MAPLE or by means of any Symbolic Program, with an immediate application to polynomials interpolation problems [4].

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