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A Study of Holders Inequality in New Spaces

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Abstract: The central attraction of this paper is to prove holders inequality in some new spaces such as Grand Lebesgue Space $L^{P)}$. Small Lebesgue space $L^{P)'}$ and Banach Lattice space X^P.

1. Introduction

The Grand Lebesgue Space $L^{p)}$, small Lebesgue Space $L^{p)'}$ and Banach lattice space X^p have many application in analysis. The grand Lebesgue spaces were introduced by Iwaniec and Sbordone and the small Lebesgue space were introduced by A. fiorenza. The sole purpose of this paper is to study the Holders inequality in these spaces. The concept of these spaces can be applied in Lorentz space as well.

Definition 1.1 (Grand Lebesgue space, $L^{P)}$): Let $\Omega \subset \Re^n$, $n \in Z^+$, be a set of Lebesgue measure. Let $|\Omega| < \infty$. The Grand Lebesgue space, denoted by $L^{P)}(\Omega)$ such that

$$||f||p^{)} = {}^{1/(P-\in)} < \infty$$

2. Holders inequality on Grand Lebesgue space, $L^{P)}$

Let $f \in L^{P'}$, $g \in L^{(P'}$ then

$$\int fgdx \le ||f||p^{)}\, ||g||_{(p'}$$

Proof:

Let $|g| = g_k$ be any decomposition with $g_k \ge 0$, $k \in N$ and let $f \in L^{P)}$. For each $k \in N$ and for each $0 < \in < p - 1$, we have

$$\int_\Omega f\, g_k dx \le \int_\Omega |f|\, g_k\, dx$$

$$\le$$

$$=$$

$$\le \in^{-1/(p-\in)}$$

Therefore,

$$\in^{-1/(p-\in)}$$

Which implies

$\leq$

$\leq$

$\leq \in^{-1(p-\in)'}$

Hence,

$\leq ||f||_{p)} \, ||g||_{(p'}$

Definition 1.3 (Small Lebesgue Space $L^{P)'}$: The Small Lebesgue Space, denoted by $L^{p)'}$, is the space of functions g in $\mathcal{M}_o$ such that

$$||g||p^{)'} = ||\psi||^{(p'} < \infty;$$

where $\mathcal{M}_o$ is the set of all measurable functions whose values lie in $[-\infty, \infty]$, finite a.e. in Ω, $\Omega \subseteq \Re^n$, $|\Omega| < \infty$.

1.4 Holders inequality in small Lebesgue space $L^{p)'}$

Let $1 < p < \infty$ and $\Omega \subseteq \Re^n$, $n \in Z^+$, $|\Omega| < \infty$. then, for $f \in L^{p)'}$ and $g \in L^{P)'}$,

$$\int_\Omega fg \, dx \leq ||f||_{p^{)}} \, ||g||_{p)'}$$

Proof:

For any $f \in L^{p)}$ and $g \in \mathcal{M}_o$ in view of 1.3

$$\int_\Omega fg \, dx \leq \int_\Omega |f| \, |g| \, dx$$

$$= \int_\Omega |f| \, \psi \, dx$$

$$= \int_\Omega |f| \, \psi \, d||_{(p'}$$

$$= ||f||_{p)} \, ||\psi||_{(p'}$$

$$= ||f||_{p)} \, ||g||_{p)'}$$

1.5 Definition (Banach Lattice)

Let (Ω, Σ, μ) be σ finite measures space. Let $L^o(\Omega)$ be the space of all extended scalar valued μ measurable function defined on Ω and μ – a.e. on finite on Ω. then, a Banach space $X = (X, ||.||_x)$ of $L^o(\Omega)$ is called a Banach lattice on (Ω, Σ, μ) if for every $x \in X$, the following implication holds:

$$y \in L^o(\Omega), |y| \leq |\mathbf{x}|\mu - \text{a.e.} \Rightarrow y \in X \text{ and } ||y||_x = ||\mathbf{x}||_x.$$

1.6 Definition (Banach Lattice)

Let X be a Banach Lattice on (Ω, Σ, μ) and let $-\infty < p < \infty$, $p \neq 0$. The space X^p consists of all $x \in L^{p'}(\Omega)$ such that $|\mathbf{x}^p| \in X$ and that

$||\mathbf{x}||_X{}^p = (||\mathbf{x}||)^{1/p} < \infty$

For the case p < 0, we assume that

$\mathbf{x} = \mathbf{x}(t) \neq 0$ for all $t \in \Omega$

1.7 Generalized Holders inequality in X^p space

Let $\mathbf{x}_i \in x^{p_i}$, $0 < p_i < \infty$, i = 1, 2, …, N. Then

$$\left\|\prod_{i=1}^{N} x_i\right\|_{x^r} \leq \prod_{i=1}^{N} ||\mathbf{x}_i||_X{}^{p_i}$$

Proof:

Let us assume that $||\mathbf{x}_i||_X{}^{p_i} \neq 0$, i = 1, 2,…, N, otherwise the assertion is trivial.

Let us put, y_i = , i = 1, 2, …, N

And $\Omega_o = \{t \in \Omega : \prod_{i=1}^{N} y_i(t) \neq 0\}$

Since = 1 and the function f(u) = exp(u) is convex, for every $t \in \Omega_o$, we have

$$\prod_{i=1}^{N} |g_i(f)|^r \quad = \exp(r \log \prod_{i=1}^{N} |g_i(t)|)$$

= exp

= exp

≤

Therefore by using lattice property and the triangle in equality, we have

$$\left\|\prod_{i=1}^{N} y_i\right\|_{x^r}^{r} \leq ||y_i|| = = 1$$

Which implies,

$$\left\|\prod_{i=1}^{N} y_i\right\|_{x^r} \leq 1$$

Hence, the assertion follows.

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