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Leverrier-Takeno Coefficients for the Characteristic Polynomial of a Matrix

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Abstract: It is shown for any $\underset{\sim}{A}_{nxn}$ an elementary method, in terms of the spur of $\underset{\sim}{A}^k, k=1,...,n$, to determine the coefficients of its characteristic polynomial.

1. Introduction

For arbitrary $\underset{\sim}{A}_{nxn}$, a matrix $(\underset{\sim}{A}-\lambda\underset{\sim}{I})$ can be constructed with a Characteristic Polynomial [1-3] given by:

$$(-1)^n \det(\underset{\sim}{A}-\lambda\underset{\sim}{I}) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + + a_{n-1}\lambda + a_n = 0 \tag{1}$$

whose roots are the eigenvalues of $\underset{\sim}{A}$. There are mentioned in [4] the different techniques to get explicitely the coefficients $a_j, j = 1,\ldots,n$, but here it is only considered the famous method of Leverrier [5] - Takeno [6] which determines a_j via the expressions [7-12]:

$$a_1 = -s_1 \quad , \quad 2!\, a_2 = (s_1)^2 - s_2 \quad , \quad 3!\, a_3 = -(s_1)^3 + 3s_1 s_2 - 2s_3 \quad ,$$

$$4!\, a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1 s_2 + 3(s_2)^2 - 6s_4 \quad , \tag{2}$$

$$5!\, a_5 = -(s_1)^5 + 10(s_1)^3 s_2 - 20(s_1)^2 s_3 + 15 s_1\left[2s_4 - (s_2)^2\right] + 20 s_2 s_3 - 24 s_5 \, ,$$

where s_j represents the spur of the j-th power of $\underset{\sim}{A}$:

$$s_j = tr\, \underset{\sim}{A}^j \quad , \quad j = 1,2,\ldots,n \tag{3}$$

that is, $s_1 = tr\, \underset{\sim}{A}$, $s_2 = tr\, \underset{\sim}{A}^2$, etc. The formulae (2) result when employing $a_1 = -s_1$ in the recurrence relation:

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + \ldots + s_{r-1} a_1 + s_r = 0 \quad , \quad r = 1,\ldots,n \, . \tag{4}$$

Furthermore, if we let $\lambda = 0$ in (1), it is obtained:

$$a_n = (-1)^n \det \underset{\sim}{A} . \tag{5}$$

Here it is shown an elementary development, alternative to (4), in order to construct expressions (2), which may be interesting when teaching Linear Algebra.

2. Leverrier-Takeno Coefficients

To reproduce (2), the basic idea consists in generating a_j by means of products $s_{i_1} s_{i_2} \ldots s_{i_k}$ such that $\sum_{c=1}^{k} i_c = j$, which will be illustrated for $n = 2, 3$ y 4, from where it will become evident how the method is applied for arbitrary *n*.

Let $n = 2$, then a_2 will have the structure:

$$a_2 = c_1 s_1 s_1 + c_2 s_2 = c_1 (s_1)^2 + c_2 s_2 , \tag{6}$$

where the quantities c_1 and c_2 are determined by selecting particular forms of $\underset{\sim}{A}$, in fact:

If $\underset{\sim}{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $\underset{\sim}{A}^2 = \underset{\sim}{A} \quad \therefore \quad s_1 = s_2 = 1$, $a_2 \overset{(5)}{=} \det \underset{\sim}{A} = 0$, so (6) implies:

$$c_1 + c_2 = 0 , \tag{7.a}$$

and for $\underset{\sim}{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have $\underset{\sim}{A}^2 = \underset{\sim}{A} = \underset{\sim}{I} \quad \therefore \quad s_1 = s_2 = 2$, $a_2 = 1$, and because (6) it results:

$$2c_1 + c_2 = \frac{1}{2} , \tag{7.b}$$

which together with (7.a) gives $c_1 = -c_2 = \frac{1}{2}$, then (6) reproduces the expression for a_2 shown in (2).

If $n = 3$, the coefficient a_3 will take the form:

$$a_3 = c_1 s_1 s_1 s_1 + c_2 s_1 s_2 + c_3 s_3 = c_1 (s_1)^3 + c_2 s_1 s_2 + c_3 s_3 , \tag{8}$$

which can be applied to the following matrices:

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \underset{\sim}{A}^3 = \underset{\sim}{A}^2 = \underset{\sim}{A} \quad \therefore \quad s_1 = s_2 = s_3 = 1 \ , \ a_3 = -\det \underset{\sim}{A} = 0 ,$$

and (8) leads to:

$$c_1 + c_2 + c_3 = 0 . \tag{9.a}$$

Similarly,

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad : \quad 4c_1 + 2c_2 + c_3 = 0, \tag{9.b}$$

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad : \quad 9c_1 + 3c_2 + c_3 = -\frac{1}{3}, \tag{9.c}$$

and the solution to the system (9.a,b,c) is $c_1 = -\frac{1}{6}$, $c_2 = \frac{1}{2}$, $c_3 = -\frac{1}{3}$, which in (8) implies the corresponding expression (2).

For $n = 4$:

$$\begin{aligned} a_4 &= c_1 s_1 s_1 s_1 s_1 + c_2 s_1 s_3 + c_3 s_2 s_2 + c_4 s_1 s_1 s_2 + c_5 s_4, \\ &= c_1 (s_1)^4 + c_2 s_1 s_3 + c_3 (s_2)^2 + c_4 (s_1)^2 s_2 + c_5 s_4, \end{aligned} \tag{10}$$

therefore:

$$\begin{aligned} \underset{\sim}{A} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &:& \quad c_1 + c_2 + c_3 + c_4 + c_5 = 0, \\ \underset{\sim}{A} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &:& \quad 8c_1 + 2(c_1 + c_3) + 4c_4 + c_5 = 0, \\ \underset{\sim}{A} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &:& \quad 2c_3 + c_5 = 0, \\ \underset{\sim}{A} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &:& \quad 27c_1 + 3(c_2 + c_3) + 9c_4 + c_5 = 0, \\ \underset{\sim}{A} &= \underset{\sim}{I}_{4x4} &:& \quad 64c_1 + 4(c_2 + c_3) + 16c_4 + c_5 = \frac{1}{4}, \end{aligned} \tag{11}$$

the system (11) admits the solution $c_1 = \frac{1}{24}$, $c_2 = \frac{1}{3}$, $c_3 = \frac{1}{8}$, $c_4 = c_5 = -\frac{1}{4}$, then (10) gives the formula for a_4 of (2).

From the previous examples it is clear the way to apply the method for any n, and in this manner to reproduce the Leverrier-Takeno relations for the coefficients of the Characteristic Polynomial of an arbitrary matrix.

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