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A Note on a Formula of Riordan Involving Harmonic Numbers

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Abstract: We employ Stirling numbers of the second kind to prove a relation of Riordan involving harmonic numbers.

Keywords: Stirling numbers, Geometric series, Riordan's identity, Harmonic numbers.

1. Introduction

We know the Riordan's relation [7]:

$$\sum_{k=1}^{n} \frac{x^k}{k} - H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(x-1)^k}{k}, \qquad \forall\, x \in \mathbb{C}, \tag{1}$$

for the harmonic numbers [6]:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{2}$$

It is usual to show (1) employing the geometric series and the binomial theorem of Newton; we observe that Agoh [1] also obtained this identity of Riordan. In Section 2, we exhibit an alternative proof of (1) via Stirling numbers [6, 7].

2. Riordan's formula

The generating function for the Stirling numbers of the second kind is given by [6]:

$$\sum_{r=k}^{\infty} \frac{z^r}{r!} S_r^{[k]} = \frac{1}{k!} (e^z - 1)^k, \tag{3}$$

with the property [3, 2]:

$$\sum_{k=1}^{n} k^{r-1} = \sum_{k=1}^{n} \binom{n}{k} (k-1)!\ S_r^{[k]}\ , \tag{4}$$

then:

$$\sum_{k=1}^{n} \frac{e^{kz}}{k} = \sum_{k=1}^{n} \frac{1}{k} \left(1 + \sum_{r=1}^{\infty} \frac{k^r z^r}{r!}\right) \overset{(2)}{=} H_n + \sum_{r=1}^{\infty} \frac{z^r}{r!} \sum_{k=1}^{n} k^{r-1}\ ,$$

that is:

$$\sum_{k=1}^{n} \frac{e^{kz}}{k} - H_n \overset{(4)}{=} \sum_{k=1}^{n} \binom{n}{k} (k-1)! \sum_{r=k}^{\infty} \frac{z^r}{r!} S_r^{[k]} \overset{(3)}{=} \sum_{k=1}^{n} \binom{n}{k} \frac{(e^z-1)^k}{k},$$

where we can use $x = e^z$ to deduce (1), q. e. d.

Our procedure is simple and shows the connection between the harmonic numbers and the Stirling numbers of the second kind [5]. For $x = 0$, the relation (1) implies the known expression of Euler [6]:

$$H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}, \tag{5}$$

Besides, from (1) is immediate the identity:

$$\sum_{n=1}^{m} H_n = (m+1) \sum_{k=1}^{m} \frac{x^k}{k} - \sum_{k=1}^{m} x^k - \sum_{k=1}^{m} \binom{m+1}{k+1} \frac{(x-1)^k}{k}, \tag{6}$$

where was applied the property $\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}$ [4]. The formula (6) with $x = 0$ and $x = 1$ implies [6]:

$$\sum_{n=1}^{m} H_n = (m+1)\, H_m - m = (m+1)\,(H_{m+1} - 1) = \sum_{k=1}^{m} \binom{m+1}{k+1} \frac{(-1)^{k+1}}{k} \tag{7}$$

Similarly from (1), it is simple to obtain for $x = 1$:

$$\sum_{n=1}^{m} n\, H_n = \frac{m}{2} \left[(m+1)H_m + \frac{1}{2}(1-m)\right] = \frac{m(m+1)}{2} \left(H_{m+1} - \frac{1}{2}\right) \tag{8}$$

3. Conclusion

Our procedure to prove (1) shows the important relationship between the harmonic numbers and the Stirling numbers of the second kind, without the participation of geometric series and the Newton's binomial theorem.

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