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A Note on Some Applications of Boyadzhiev's Formula

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Abstract: In this paper, we employ an expression of Boyadzhiev to give elementary proofs of the identities for harmonic numbers obtained by Paule-Schneider via the Mathematica package *Sigma*.

Keywords: Boyadzhiev's identity, harmonic numbers, Paule-Schneider's relations.

1. Introduction

Boyadzhiev [2, 6] evaluated several binomial transforms by using Euler's transform for power series and obtained various binomial identities involving power sums with harmonic numbers. Boyadzhiev's Formula for harmonic numbers is as follows:

$$\sum_{k=1}^{n} \binom{n}{k} H_k \, \lambda^{n-k} \, \mu^k = (\lambda+\mu)^n \, H_n - \sum_{j=1}^{n} \frac{\lambda^j}{j} \, (\lambda+\mu)^{n-j} \qquad (1)$$

Paule-Schneider [8] used the Mathematica package *Sigma* [11] and the Zeilberger's method [9, 13] to find the following identities [5] for harmonic numbers [1, 3]:

$$\sum_{j=1}^{n} H_j \binom{n}{j} = 2^n \left[H_n - \sum_{j=1}^{n} \frac{1}{j\, 2^j} \right], \qquad H_n \equiv \sum_{k=1}^{n} \frac{1}{k}, \qquad (2)$$

$$\sum_{j=1}^{n} j \, H_j \binom{n}{j} = -\frac{1}{2} + 2^{n-1} \left[1 + n\, H_n - n \sum_{j=1}^{n} \frac{1}{j\, 2^j} \right]. \qquad (3)$$

Here we employ (1) to exhibit elementary proofs of these interesting identities.

2. Paule-Schneider's relations

In (1) we can use the values $\lambda = \mu = 1$ to deduce (2):

$$\sum_{k=1}^{n} \binom{n}{k} H_k = 2^n \left(H_n - \sum_{j=1}^{n} \frac{1}{j\,2^j}\right) \overset{[5]}{=} \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \frac{2^{n-j}}{j}, \qquad n \geq 1. \tag{4}$$

The identity (1) with $\lambda = -\mu = 1$ implies the relation [4, 10]:

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} H_k = \frac{1}{n}, \qquad n \geq 1, \tag{5}$$

which is a particular case of the following formula [12] ($H_0 = 0$):

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_{k+m} = \frac{1}{n\binom{m+n}{n}}, \qquad n \geq 1, \quad m \geq 0. \tag{6}$$

Besides, we obtain (6) from [7]:

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_{k+s}(x) = \frac{(n-1)!}{(x+s+1)_n}, \tag{7}$$

if $x = 0$.

Now we apply $\frac{d}{d\mu}$ to (1), thus:

$$\sum_{k=1}^{n} k \binom{n}{k} H_k\, \lambda^{n-k} \mu^{k-1} = n\,(\lambda+\mu)^{n-1} H_n - \sum_{j=1}^{n-1} \frac{n-j}{j} \lambda^j (\lambda+\mu)^{n-1-j}, \tag{8}$$

which for $\lambda = \mu = 1$ gives:

$$\sum_{k=1}^{n} k \binom{n}{k} H_k = 2^{n-1}\left(n\,H_n - \sum_{j=1}^{n} \frac{n-j}{j\,2^j}\right) = n\,2^{n-1}\left(H_n - \sum_{j=1}^{n} \frac{1}{j\,2^j}\right) + 2^{n-1} \sum_{j=1}^{n} \frac{1}{2^j}, \tag{9}$$

but $\sum_{j=1}^{n} \frac{1}{2^j} = 1 - \frac{1}{2^n}$, then (9) implies (3), that is [5]:

$$\sum_{k=1}^{n} k \binom{n}{k} H_k = 2^{n-1} - \frac{1}{2} + n \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \frac{2^{n-1-j}}{j}. \tag{10}$$

From (9) with $\lambda = -\mu = 1$ is immediate the identity:

$$\sum_{k=1}^{n} (-1)^{k+1}\, k \binom{n}{k} H_k = \begin{cases} 1, & n = 1, \\ \frac{1}{1-n}, & n \geq 2. \end{cases} \tag{11}$$

3. Conclusion

The approach in this paper shows that the identities (2) and (3) obtained by Paule-Schneider [8] can be deduced without the Mathematica package *Sigma* and the Zeilberger algorithm [9, 13].

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