

Two Higher Order Iterative Methods for Solving Nonlinear Equations

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Received: Jan 5, 2018

Revised: Jan 20, 2018

Accepted: Jan 25, 2018

Abstract: The main purpose of this paper is to derive two higher order iterative methods for solving nonlinear equations as variants of Mir, Ayub and Rafiq method. These methods are free from higher order derivatives. We obtain these methods by amalgamating Mir, Ayub and Rafiq method with standard secant method and modified secant method given by Amat and Busquier. The order of convergence of new variants are four and six. Also, numerical examples are given to compare the performance of newly introduced methods with the similar existing methods.

2010 AMS Subject Classification: 65H05

Keywords: Newton's method, secant method, iterative method, nonlinear equation, order of convergence

1. Introduction

Finding the solutions of single variable nonlinear equations efficiently is an important consideration in numerical analysis and has wide range of applications in all areas of science and engineering. Finding analytic solutions of such equations is not always possible and most widely used classical numerical methods for solving such equations are the Newton method [3]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0 \quad (1)$$

and secant method [3]

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \quad n \geq 1 \quad (2)$$

The order of convergence of the Newton method is 2 and the secant method is 1.618. Secant method (2) can be considered as a simplification of Newton's method (1), where $f'(x)$ is replaced by

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The main limitation of this method with respect to the Newton method is of order since the above expression is not optimal solution of $f'(x_n)$. In order to avoid this problem, Amat and Basquier [2] provide following generalization of secant method:

$$x_{n+1} = x_n - A_n^{-1} f(x_n) \quad (3)$$

where

$$A_n = [y_n, x_n; f] = \frac{f(x_n) - f(y_n)}{x_n - y_n}, \quad "y_n = x_n + \delta_n (x_{n-1} - x_n)", \quad \delta_n \leq \left| O(e_n)^{\frac{3}{2}} \right|.$$

The order of convergence of this method is same as Newton's method.

In the last two decades, some researchers have developed several numerical methods based on integral approximations (see [1, 4, 6, 7, 8, 9, 10, 11, 13, 15, 16, 17, and 18]).

In [5], Hassanov, Ivanov and Nedzhibov suggested a new iterative method

$$x_{n+1} = x_n - \frac{6 f(x_n)}{f'(x_n) + 4 f' \left(\frac{x_n + x_n^*}{2} \right) + f'(x_n^*)} \quad (4)$$

$$\text{where } x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$$

by approximating the integral in Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^x f'(\lambda) d\lambda$$

using the Simpson rule. Combining this method with the quadratically convergent method due to Kanwar, Kukreja and Singh [14],

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(x_n) + p \sqrt{(f'(x_n))^2 + 4 f^2(x_n)}} \quad (5)$$

Mir, Ayub and Rafiq [15] obtained a new third order method

$$x_{n+1} = x_n - \frac{6 f(x_n)}{f'(x_n) + 4 f' \left(\frac{x_n^* + x_n}{2} \right) + f'(x_n^*)} \quad (6)$$

$$\text{where } x_{n+1}^* = x_n - \frac{2 f(x_n)}{f'(x_n) + p \sqrt{(f'(x_n))^2 + 4 f^2(x_n)}}$$

for solving nonlinear equation $f(x) = 0$, where p is chosen as positive or negative sign so as to make the denominator largest in magnitude to avoid numerical difficulties. Our aim, in this paper, is to find two variants of method (6) by combining the iteration of this method with secant method and modified secant method. The order of convergence of new methods will be 4 and 6.

2. Description of Methods

In this section, we propose two variants of Mir, Ayub and Rafiq method (6). At first, we propose a method by performing the iterations of Mir, Ayub and Rafiq method alternately with secant method. The corresponding method becomes

$$x_{n+1} = \bar{x}_n - \frac{\bar{x}_n - x_n}{f(\bar{x}_n) - f(x_n)} f(\bar{x}_n) \tag{7}$$

where
$$\bar{x}_n = x_n - \frac{6 f(x_n)}{f'(x_n) + 4 f'(\frac{x_{n+1}^* + x_n}{2}) + f'(x_{n+1}^*)} \tag{8}$$

with
$$x_{n+1}^* = x_n - \frac{2 f(x_n)}{f'(x_n) + p \sqrt{(f'^2(x_n) + 4 f^2(x_n))}} \tag{9}$$

In this method, p is chosen as positive or negative sign so as to make the denominator largest in magnitude to avoid numerical difficulties.

Next, the iterations of Mir, Ayub and Rafiq method (6) are performed alternately with modified secant method (3). Then the corresponding method becomes

$$x_{n+1}^* = x_n - \frac{2 f(x_n)}{f'(x_n) + p \sqrt{(f'^2(x_n) + 4 f^2(x_n))}}$$

$$\bar{x}_n = x_n - \frac{6 f(x_n)}{f'(x_n) + 4 f'(\frac{x_{n+1}^* + x_n}{2}) + f'(x_{n+1}^*)}$$

$$x_{n+1} = \bar{x}_n - A_n^{-1} f(\bar{x}_n) \tag{10}$$

where $A_n = [y_n, \bar{x}_n; f] = \frac{f(\bar{x}_n) - f(y_n)}{\bar{x}_n - y_n}$

with $y_n = \bar{x}_n + \delta_n (x_n - \bar{x}_n), \quad \delta_n \leq \left| O(e_n)^{\frac{3}{2}} \right|.$

3. Analysis of Convergence

For the convergence analysis of method (7) - (9), we prove following result:

Theorem 1. Let f be a function having sufficient number of derivatives in a neighborhood of α , which is a simple root of equation $f(x) = 0$. Then, the iterative method (7)-(9) is of order 4 for $p = 1$.

Proof. Let e_n and \bar{e}_n be the errors in x_n and \bar{x}_n , respectively, that is, $x_n = \alpha + e_n$ and $\bar{x}_n = \alpha + \bar{e}_n$. Denote $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$, $k = 2, 3, 4, \dots$. It was shown by Mir, Ayub and Rafiq in [15] that the error equation of (8) is given by

$$\bar{e}_n = c_2^2 e_n^3 + O(e_n^4) \quad (11)$$

Here,

$$\begin{aligned} \bar{x}_n - x_n &= (\alpha + \bar{e}_n) - (\alpha + e_n) \\ &= \bar{e}_n - e_n \\ &= c_2^2 e_n^3 - e_n + O(e_n^4). \end{aligned}$$

By Taylor's expansion, we get

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) \\ &= f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \end{aligned}$$

and using (11), we obtain

$$\begin{aligned} f(\bar{x}_n) &= f(\alpha + \bar{e}_n) \\ &= f'(\alpha)[c_2^2 e_n^3 + O(e_n^6)] \end{aligned}$$

Thus, we get

$$\begin{aligned} f(\bar{x}_n) - f(x_n) &= f'(\alpha)[c_2^2 e_n^3 + O(e_n^6)] - f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \\ &= -f'(\alpha)e_n [1 + c_2 e_n + (c_3 - c_2^2)e_n^2 + O(e_n^3)] \end{aligned}$$

and

$$\begin{aligned} \frac{(\bar{x}_n - x_n)f(\bar{x}_n)}{f(\bar{x}_n) - f(x_n)} &= \frac{[c_2^2 e_n^3 - e_n + O(e_n^4)] f'(\alpha)[c_2^2 e_n^3 + O(e_n^6)]}{-f'(\alpha)e_n [1 + c_2 e_n + (c_3 - c_2^2)e_n^2 + O(e_n^3)]} \\ &= [c_2^2 e_n^3 + O(e_n^5)][1 + c_2 e_n + (c_3 - c_2^2)e_n^2 + O(e_n^3)]^{-1} \\ &= c_2^2 e_n^3 - c_2^3 e_n^4 + O(e_n^5) \end{aligned}$$

Thus, the error equation in (7) is given by

$$\begin{aligned} e_{n+1} &= \bar{e}_n - c_2^2 e_n^3 + c_2^3 e_n^4 + O(e_n^5) \\ &= c_2^2 e_n^3 + O(e_n^4) - c_2^2 e_n^3 + c_2^3 e_n^4 + O(e_n^5) \\ &= A e_n^4 + O(e_n^5), \end{aligned}$$

where A is some constant. The assertion is, therefore, proved.

For the convergence analysis of method (10), we prove following result:

Theorem 2. Let f be a function having sufficient number of derivatives in a neighborhood of α , which is a simple root of equation $f(x) = 0$. Then, the iterative method (10) is of order at least 6 for $p = 1$ and suitable choice of δ .

Proof. We prove this theorem in the line of proof of Theorem 1. The error \bar{e}_n in \bar{x}_n is given by

$$\bar{e}_n = c_2^2 e_n^3 + O(e_n^4).$$

Since

$$y_n = \bar{x}_n + \delta_n (x_n - \bar{x}_n) = \bar{x}_n + a_n, \quad \text{where } a_n =$$

$$\delta_n (x_n - \bar{x}_n),$$

by using Taylor expansion, we get

$$f(y_n) = f(\bar{x}_n + a_n) = f(\bar{x}_n) + a_n f'(\bar{x}_n) + \frac{a_n^2}{2} f''(\bar{x}_n) + \dots.$$

Also,

$$\begin{aligned} A_n &= \frac{f(\bar{x}_n) - f(y_n)}{\bar{x}_n - y_n} \\ &= \frac{a_n f'(\bar{x}_n) + \frac{a_n^2}{2} f''(\bar{x}_n) + \dots}{a_n} \\ &= f'(\bar{x}_n) + \frac{a_n}{2} f''(\bar{x}_n) + \dots \\ &= f'(\bar{x}_n) + O(a_n). \end{aligned}$$

Thus, from (10), we have

$$\begin{aligned} x_{n+1} &= \bar{x}_n - \frac{f(\bar{x}_n)}{A_n} \\ &= \bar{x}_n - \frac{f(\bar{x}_n)}{f'(\bar{x}_n) + O(a_n)} \\ &= \bar{x}_n - \frac{f(\bar{x}_n)}{f'(\bar{x}_n)} [1 + O(a_n)]^{-1} \\ &= \bar{x}_n - \frac{f(\bar{x}_n)}{f'(\bar{x}_n)} [1 + O(a_n) + O(a_n^2) + \dots] \\ &= \bar{x}_n - \frac{f(\bar{x}_n)}{f'(\bar{x}_n)} + O(a_n). \end{aligned}$$

Thus, the method (10) can be written as

$$x_{n+1} = \bar{x}_n - \frac{f(\bar{x}_n)}{f'(\bar{x}_n)} + O(\delta_n |x_n - \bar{x}_n|) \tag{11}$$

From (11), we have

$$\begin{aligned} e_{n+1} &= \bar{e}_n - \frac{f(\alpha + \bar{e}_n)}{f'(\alpha + \bar{e}_n)} + O(\delta_n |\alpha + e_n - \alpha - \bar{e}_n|) \\ &= \bar{e}_n - [\bar{e}_n + c_2 \bar{e}_n^2 + c_3 \bar{e}_n^3 + O(\bar{e}_n^4)] [1 + 2c_2 \bar{e}_n + 3c_3 \bar{e}_n^2 + O(\bar{e}_n^3)]^{-1} \\ &\quad + O(\delta_n |e_n - \bar{e}_n|) \end{aligned}$$

$$\begin{aligned}
&= \bar{e}_n - [\bar{e}_n + c_2 \bar{e}_n^{-2} - 2c_3 \bar{e}_n^{-2} + O(\bar{e}_n^{-3})] + O(\delta_n |e_n - \bar{e}_n|) \\
&= c_2 \bar{e}_n^{-2} + O(\bar{e}_n^{-3}) + O(\delta_n |e_n - \bar{e}_n|) \\
&= c_2 (c_2^2 e_n^3)^2 + O(\delta_n |e_n - c_2^2 e_n^3|) + \dots \\
&= c_2^5 e_n^6 + O(\delta_n |e_n - c_2^2 e_n^3|) + \dots
\end{aligned} \tag{12}$$

Since $\delta_n \leq |O(\bar{e}_n^{-3})|$, so if we assume that $\delta_n \leq |O(\bar{e}_n^{-2})|$, that is, $\delta_n \leq |O(e_n)^6|$, then the order of the method (10) is at least 6.

Remark 1. In practice, we do not have any information about the solution. In this situation, we use the same strategy as the Amat and Basquier indicated in [2] to obtain δ_n . The possible strategy to obtain $\{\delta_n\}$ can be

$$\begin{aligned}
\delta_0 &= O(10^{-k}) \leq O(e_n)^{\frac{3}{2}} \\
\delta_n &= O(\delta_0^{2^n}),
\end{aligned}$$

where k is an integer such that

$$O(10^{-k}) \leq |f(\alpha) - f(x_0)| \leq |f(x_0)| \leq O(|\alpha - x_0|).$$

4. Numerical Examples

In this section, we shall compare the performance of newly introduced methods (7) and (10) with the some existing methods. To avoid the numerical difficulties in newly introduced methods, we take p equal to 1 or -1 so as to make the denominator largest in magnitude. For the comparison, we use Matlab Software and stopping criteria $|x_{n+1} - x_n| < (10)^{-12}$ or $|f(x_{n+1})| < (10)^{-14}$ for the iterative process of our results.

Example 1. We apply methods (7), and (10) on the nonlinear equation

$$x^6 - x - 1 = 0 \tag{13}$$

From the intermediate value theorem, it is clear that one of the simple roots of (13) lies in the interval (1, 2). Taking initial approximation as $x_0 = 1$, Table 1 shows the iterations of the Newton method, Mir, Ayub and Rafiq method (6), and introducing methods (7) and (10).

Table 1: Comparison of different methods.

Method	n	x_n	$ x_n - x_{n-1} $	$ f(x_n) $
Newton's method	1	1.2000000000000000	0.2000000000000000	0.7859839999999999
	2	1.143575842503044	0.056424157496956	0.093031957363097
	3	1.134909462242086	0.008666380260958	0.001907397172335
	4	1.134724221386558	0.000185240855528	0.000000853719439
	5	1.134724138401536	0.000000082985022	0.000000000000171
	6	1.134724138401519	0.000000000000017	0.000000000000001
Mir, Ayub and Rafiq method (p=1)	1	1.114331079664285	0.114331079664285	0.199698426927704
	2	1.134671170083933	0.020340090419648	0.000544848619648
	3	1.134724138400651	0.000052968316718	0.0000000000008935
	4	1.134724138401519	0.000000000000868	0.000000000000001
Present method (7)	1	1.142859996170415	0.142859996170415	0.085360616958645
	2	1.134724198032685	0.008135798137730	0.000000613463385
	3	1.134724138401519	0.000000059631166	0.000000000000001
Present method (10) ($\delta_0 = 0.15$)	1	1.134888830552744	0.134888830552744	0.001694966368938
	2	1.134724138401519	0.000164692151225	0.000000000000003

Example 2. We apply methods (7) and (10) on the nonlinear equation

$$\sin^2 x - x^2 + 1 = 0 \tag{14}$$

From the intermediate value theorem, it is clear that one of the simple roots of (14) lies in the interval (1, 2). Taking initial approximation as $x_0 = 2$, Table 2 shows the iterations of the Kanwar, Kukreja, and Singh method (5), Mir, Ayub and Rafiq method (6), and introducing methods (7) and (10) when we apply on (14).

Table 2: Comparison of different methods.

Method	n	x_n	$ x_n - x_{n-1} $	$ f(x_n) $
Kunar, Kukraia and Singh method	1	1.611941042164389	0.388058957835611	0.600045855942974
	2	1.436057898886696	0.175883143277692	0.080307137216431
	3	1.405269547988669	0.030788350898027	0.001932292231027
	4	1.404492122350878	0.000777425637791	0.000001177029015
	5	1.404491648215517	0.000000474135360	0.000000000000437
	6	1.404491648215341	0.000000000000176	0.000000000000000
Mir, Ayub and Rafiq method (p=1)	1	1.463929732530918	0.536070267469082	0.154467321201545
	2	1.404614185557379	0.059315546973539	0.000304224828280
	3	1.404491648216471	0.000122537340908	0.000000000002804
	4	1.404491648215341	0.000000000001130	0.000000000000000
Present method (7)	1	1.422910813234791	0.577089186765209	0.046386337604170
	2	1.404491702133259	0.018419111101532	0.000000133849771
	3	1.404491648215341	0.000000053917918	0.000000000000000
Present method (10) ($\delta_0 = 0.1$)	1	1.404751654708388	0.595248345291612	0.000645590555316
	2	1.404491648215341	0.000260006493047	0.000000000000000

5. Conclusion

From the above comparison tables, we observe that newly introduced methods are easily compete with existing Newton's method, Kanwar, Kukreja, and Singh's method (5) and Mir, Ayub and Rafiq's method (6). For the suitable choice of the values of p and δ_0 , newly introduced methods (7) and (10) provide better results.

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