Journal of the Institute of Engineering
Volume 17, No. 1
ISSN: 1810-3383
Published: April 2023

Burgers' Equation and Traffic Flow

Amrit Sharma [a], Durga Jung K.C. [b]

[a,b]*Central Department of Mathematics, IOST, Tribhuvan University, Kathmandu, Nepal*
Corresponding Author: [b]durgajkc@gmail.com

Received: 2022-09-27 Revised:2023-03-21 Accepted: 2023-03-22

Abstract:
Burgers' equation is a well-known partial differential equation that arises in many areas such as fluid mechanics, non-linear acoustics, gas dynamics, and traffic flow. This paper aims to discuss Burgers' equation and its applications. We begin with the historical development of Burgers' equation. We mainly focus on the study of traffic flow models where we use the inviscid version of Burger' equation to model the flow. To this end, we discuss the classification of traffic flow models and some recent development of traffic flow models.

Keywords: Burgers' equation, Inviscid burgers equation, Traffic flow, Traffic flow models

1.Introduction

Linear and non-linear partial differential equations (PDEs) are used to model different physical phenomena in science and engineering. Many simple and complex PDEs are developed to model real life problems which are also related to the engineering system. Different physical processes which play very significant roles in development, design and analysis of engineering system are explained by the PDEs. For various physical model we must have knowledge of flow of fluids. Water system of a city, blood carried throughout our bodies by arteries and veins, flow of car and vehicles on a road and gas dynamic are some physical processes which work on flow of fluids. In applied mathematics, Burgers' equation occurs in the study of turbulence, gas-fluid dynamics, heat conduction and traffic flow problems. Our attention in this paper is to study traffic flow problems in the light of traffic models that are derived from Burgers' Equation. One- way traffic flow is modelled by using inviscid Burgers' equation. We first begin with Burgers' equation.

2. Burgers' Equation

Burgers' equation was first introduced by Henary Bateman [5] while describing the motion of viscous fluid in 1915. Precisely, Burgers' equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \qquad \ldots(2.1)$$

where u, x, t and ν are the velocity, spatial coordinate, time and kinematic viscosity respectively. If $\nu \neq 0$, then the equation is called viscous Burgers' equation and

when $\nu = 0$, it is called inviscid Burgers' equation. The Burgers' equation is the non-linear convection-diffusion equation which has convective term $u\frac{\partial u}{\partial x}$ and diffusive term $\nu\frac{\partial^2 u}{\partial x^2}$.

Burger developed this equation in 1948 to study the turbulence described by the interaction of two opposite effects of convection and diffusion. The Burgers' equation is parabolic with the viscous term and if the viscous term is not included then the equation is hyperbolic. The equation (2.1) is a non-linear equation very similar to the Navier-Stokes equation.

The Burgers' equation is named for Johannes Martinus Burgers' (1895-1981). Eberhard Hopf in 1950 and Julian David Cole in 1951 independently introduced a transformation which convert Burgers' equation into a linear heat equation and solved exactly for an arbitrary initial condition [9]. This transformation is known as Cole-Hopf transformation. The Hopf-Cole transformation is

$$u(x,t) = -2\nu \frac{w_x}{w}$$

where w is a dependent variable satisfies the famous heat equation.

$$\frac{\partial \mathrm{w}}{\partial \mathrm{t}} = \nu \frac{\partial^2 w}{\partial \mathrm{x}^2}$$

In 1972, Benton and Platzman [6] classified 35 distinct solutions of one-dimensional Burgers' equation with different initial conditions in tabular form. Engelberg[12] in 1998 studied the existence and the shock profile of the Burgers' equation. He used Hopf-Cole transformation to show when shock like profiles exist. He also studied the decay of perturbations of shock-like profiles of the Burgers' equation. Bogaevsky [8] found in 2004 that the matter accumulates in the shock discontinuities in inviscid solutions of the forced Burgers' equation. Taku Ohwada [18] in 2009, proposed a numerical method to solve the Burgers' equation by using diffusion equation. He also used Hopf-Cole transformation for an alternative basis of shock capturing scheme. The exact solution is given by:

$$u(x,t) = \left[1 + \exp\left[\frac{2x-t}{4\nu}\right] \frac{1+\mathrm{erf}\left[\frac{x}{\sqrt{2\nu t}}\right]}{1-\mathrm{erf}\left[\frac{x-t}{\sqrt{2\nu t}}\right]}\right]^{-1}$$

Where, $\mathrm{erf}[x] = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-y^2)dy$.

Abazari and Borhanifar [1] applied differential transformation method to solve Burgers' and coupled Burgers' equation in 2010. They found that differential transformation method is exact and easy to apply. This method reduces the computational difficulties of the other methods. Khalifa et. al [15] in 2011 used spectral method to solve non-linear Burgers' equation. They proposed Legendre polynomials as a basis for the space of solutions and it gives better results than others. Adegboyegun[2] in 2013 applied Adomian's decomposition method to approximate solution of Burgers' equation. He obtained an explicit solution for Burgers' equation with low kinematic viscosity which was not exist before that. Ali Kurt et. al [16] discovered approximate analytical solution of time conformable fractional Burgers' equation by using a homotopy analysis method in 2015. Ucar et. al [21] in 2017 found the numerical solutions of the modified Burgers' equation by finite difference method. The one-dimensional generalized Burgers' equation

$$U_t + U^p U_x - \nu\, U_{xx} = 0$$

when $p = 2$ the equation is known as the modified Burgers' equation. They also discovered that the error norms L_2 and L_∞ are sufficiently small. We next discuss an application of Burgers' equation in traffic flow and its behaviour.

3. Traffic Flow and its Behaviour

When we observe traffic flow from a distance, the flow of heavy traffic appears to be a fluid stream. As a result, by viewing traffic as an essentially one-dimensional compressible fluid, a macroscopic theory of traffic may be created using hydrodynamic theory of fluids. In traffic flow individual vehicle behavior is ignored, and one is only concerned with the behavior of a group of vehicles. To handle vehicle number conservation on a route, the earliest traffic flow models began by developing the balance equation. In fact, the law of conservation of the number of vehicles on the road must be satisfied by all traffic flow models and theories.

4. Traffic Models

Traffic models are used to understand the traffic behavior and to develop efficient traffic control plans. Different traffic conditions occur due to jams, accidents and sudden changes in traffic. Drivers' reaction is due to forward conditions, which changes density and velocity of vehicles. Traffic flow models can be classified into macroscopic, microscopic and mesoscopic models. The macroscopic models are based on the assumption that there are large number of vehicles on the road, so these models are based on the aggregate behavior of the vehicles. Macroscopic models formulate the relationships between speed, density and flow of traffic. Macroscopic models describe the traffic as continuum flow. The microscopic traffic models describe all the components of traffic flow in details and interactions between them. Microscopic models simulate single vehicle-driver unit. To describe the traffic, the microscopic models generally use the microscopic variables such as velocity, position and acceleration of individual vehicles. Microscopic models describe the behavior of individual vehicle. Mesoscopic traffic flow models were developed to fill the gap between macroscopic and microscopic models. It simulates models by calculating some elements macroscopically and some microscopically. Microscopic models forecast the traffic in more details than macroscopic models. The microscopic models are suitable for control system for road vehicles that automatically adjust the speed to maintain a safe distance from vehicle ahead. These models are used in different applications where the predictions of individual vehicles are necessary. But in the applications where fast computations are important rather than details, the macroscopic models are preferable. The applications such as management of traffic for large scale and evacuation planning require more realistic macroscopic models. The hybrid models are used in the applications which require detail information to cover a small area and fast computation to estimate over a long-time horizon. Few traffic flow models are listed as follows [19]:

1. Equilibrium model:
   $\rho_t + (\rho v)_x = 0.$
2. Isotropic inviscid non-equilibrium model:
   $\rho_t + (\rho v)_x = 0.$
   $v_t + v v_x + \frac{C^2(\rho)}{\rho}\rho_x = \frac{V_e(\rho)-v}{T}.$
3. Anisotropic inviscid non-equilibrium model:
   $\rho_t + (\rho v)_x = 0.$
   $v_t + \big(v - C(\rho)\big)v_x = 0.$
4. Isotropic viscous non-equilibrium model:
   $\rho_t + (\rho v)_x = 0.$
   $v_t + v v_x + \frac{C^2(\rho)}{\rho}\rho_x = \frac{V_e(\rho)-v}{T} + \mu v_{xx}.$
5. Anisotropic viscous non-equilibrium model:
   $\rho_t + (\rho v)_x = 0.$
   $v_t + \big(v + 2\beta C(\rho)\big)v_x + \frac{C^2(\rho)}{\rho}\rho_x = \frac{V_e(\rho)-v}{T} + \mu(\rho)v_{xx}.$

In this paper we study only those traffic models which evolved Burgers' equation.

5. Burgers Equation in Traffic Flow Models

Traffic flow models have been developed since the beginning of the twentieth century. The first traffic model was presented by Bruce Greenshields [13] in 1934 at thirteenth annual meeting of highway research board. After the development of the first model researchers developed many other models and simulation tools. They included different dynamics in the models and applied for the predictions of traffic conditions. The first model of traffic flow presented by Greenshields is based on the assumption that there is some relation between the distance between vehicles and their velocity [13]. The others models developed after that also based on the same assumption. A powerful relationship between fluid dynamics and traffic flow models is represented by the Burgers' equation. Greenshields in 1935 presented the traffic flow model which relates density and speed of the vehicles at fourteenth annual meeting of highway research board. This model first time formed Burgers' equation in the theory of traffic flow models. In this model Greenshields assumed that velocity v only depends on density ρ. If density is minimum i.e., $\rho = 0$ or the road is empty, then the velocity is maximum i.e., $v = v_{max}$. If the density is maximum i.e $\rho = \rho_{max}$ or in heavy traffic, the velocity is zero i.e. $v = 0$. The density velocity relation is given by,

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) \qquad \dots (3.1)$$

The kinematic wave model which is the preliminary version of macroscopic model developed by Lighthill and Whitham [17] in 1955 and Richards in 1956 is formed Burgers' equation. They used density velocity relationship proposed by Greenshields. For traffic flow model traffic flow of vehicles on a road with only one lane considered. Let density of the vehicles (vehicles per kilometer) be $\rho(x,t)$ in $x \in \mathbb{R}$ and $t \geq 0$.

Then the number of vehicles which are in the interval (x_1, x_2) at time t is

$$\int_{x_1}^{x_2} \rho(x,t)dx$$

The velocity of the vehicles in x at time t is $v(x,t)$. Then, the numbers of vehicles which pass through x at time t is

$$\rho(x,t)v(x,t)$$

From the conservation law, the number of vehicles in the interval (x_1, x_2) changes according to the number of vehicles which enter or leave this interval is given by

$$\frac{d}{dx}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t)$$

Integrate above equation with respect to time and assume that ρ and v are regular functions, then

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_t\, \rho(x,t)dx\, dt = \int_{t_1}^{t_2}(\rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t))dt$$

$$= -\int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_x\big(\rho(x,t)v(x,t)\big)dx\, dt$$

Here, $x_1,\ x_2 \in \mathbb{R}$ and $t_1, t_2 > 0$ are arbitrary, so we can conclude

$$\rho_t + (\rho v)_x = 0, \qquad x \in \mathbb{R}, \quad t > 0 \qquad \ldots(3.2)$$

which is a partial differential equation, let the initial condition be

$$\rho(x,0) = \rho_0(x), \qquad x \in \mathbb{R} \qquad \ldots(3.3)$$

Then from relation (3.1),

$$\rho_t + \left[v_{max}\rho\left(1 - \frac{\rho}{\rho_{max}}\right)\right]_x = 0, \quad x \in \mathbb{R}, \quad t > 0 \qquad \ldots(3.4)$$

The above equation is conservation law for the number of vehicles. Integrating (3.4) over $x \in \mathbb{R}$, gives

$$\frac{d}{dt}\int_{\mathbb{R}} \rho\,(x,t)dx = \int_{\mathbb{R}}\left[v_{max}\rho(x,t)\left(1 - \frac{\rho(x,t)}{\rho_{max}}\right)\right]dx = 0$$

Therefore, the number of vehicles in $\mathbb{R}$ is a constant for all $t \geq 0$. The equation (3.4) can be simplified by making it dimensionless form. Let L and τ be the length and time respectively such that $v_{max} = \frac{L}{\tau}$. Introducing

$$x_s = \frac{x}{L}, \qquad t_s = \frac{t}{\tau}, \qquad u = 1 - \frac{2\rho}{\rho_{max}}$$

Then ,

$$\rho = \frac{\rho_{max}}{2}\,(1-u)$$

So,

$$\partial_t\rho = \frac{1}{\tau}\partial_{t_s}\left[\frac{\rho_{max}}{2}\,(1-u)\right] = -\frac{\rho_{max}}{2\tau}\partial_{t_s}u$$

and

$$\partial_x\left[v_{max}\rho\left(1 - \frac{\rho}{\rho_{max}}\right)\right] = \frac{1}{L}\partial_{x_s}\left[v_{max}\frac{\rho_{max}}{2}\,(-u)\frac{1}{2}(+u)\right] = -\frac{\rho_{max}}{2\tau}\partial_{x_s}\left(\frac{u^2}{2}\right)$$

By using (x,t) instead of $(x_s,\ t_s)$, the equations (3.3) and (3.4) can be written as

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad x \in \mathbb{R}, \quad t > 0 \quad \ldots(3.5)$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R} \qquad \ldots(3.6)$$

with $u_0(x) = 1 - \frac{2\rho_0}{\rho_{max}}$. If the road is empty $(\rho = 0)$ then $u = 1$ and in a tailback $(\rho = \rho_{max})$ then $u = -1$.

So, the Lighthill-Whitham-Richards Model (LWR) is

$$\rho_t + (\rho v(\rho)_x) = 0, \qquad v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right), \quad 0 \leq \rho \leq \rho_{max}$$

which can be simplified by the transformation $u = 1 - \frac{2\rho}{\rho_{max}}$ to obtain

$$u_t + \frac{1}{2}(u^2)_x = 0$$

which is Inviscid Burgers Equation.

The LWR model used the continuity equation and speed-density relation to present the traffic flow. This model assumes traffic flow in equilibrium. That is, it assumes that the speed and density values at any point in the flow at any time are according to the equilibrium relation [20]. This model has received attention of many researchers and critical analysis. The main analysis is that the vehicles are achieve new speeds when the density increases or decrease, that means acceleration or deceleration is infinite. This problem has addressed by higher order macroscopic models and generalized LWR models.

Greenberg in 1959 assumed a logarithmic relationship between speed and density, that means it is assumed that the velocity of the vehicles can be very large for low densities. The main feature of this model is that this model can be derived analytically, but the main drawback of this model is that as density tends to zero, speed tends to infinity. This shows the inability of the model to predict the speeds at lower densities. The Greenberg Model is

$$\rho_t + (\rho v(\rho)_x) = 0, \qquad v(\rho) = v_{max} \ln \frac{\rho}{\rho_{max}}, \quad 0 \le \rho \le \rho_{max}$$

This implies

$$\rho_t - v_{max}(\rho \ln \rho)_x = 0$$

To address the issue of infinite acceleration or deceleration in the LWR model, Payne proposed a high order traffic flow model in 1971 which is based on car flowing theory and traffic adjustments are due to driver response. Whitham proposed a similar traffic flow model in 1974, which is known as Payne-Whitham (PW) model. It is based on the assumption that all vehicles have similar behavior, but in reality, the behavior of all vehicles is not same, so this model can lead to unrealistic results. The Payne-Whitham (PW) Model is

$$\rho_t + (\rho v)_x = 0,$$
$$(\rho v)_t + [\rho v^2 + p(\rho)]_x = 0$$

But in 1994, Carlos F. Daganzo [11] showed that the high order modifications lead to fundamentally flawed model structure. These modifications can make the things worse. The PW model mimics the traffic flow as the flow of gas particles. Carlos also showed that the representation of traffic flow as a fluid in higher order models are not reasonable and lead to unrealistic results [11]. One of the macroscopic phenomena, traffic hysteresis was discussed by Zhang [25] in 1999. He proposed that if different parameters of traffic flow should be distinguished in obtaining fundamental relationships, only then the phase transition from one phase to other can correctly identified. Chowdhury et. al [10] in 2000 found that, from a far distance, the traffic flow can be considered as an one dimensional compressible fluid. To correct the drawbacks of the PW model, Aw and Rascle proposed a modified model in 2000, which referred as AR model [4]. The Aw-Rascle Model is given by

$$\rho_t + (\rho v)_x = 0,$$
$$[\rho v + \rho p(\rho)]_t + [\rho v^2 + \rho v p(\rho)]_x = 0$$

The AR model avoids the gas like behavior, but it fails that in real life situations the velocity and density have limits. Aw et. al [3] in 2002 showed that Aw-Rascle model has been derived from microscopic model with a scaling in space and time for which the density and the velocity remain fixed. Xue Yu [23] in 2002, extended optimal velocity model of traffic flow to take into account the relative velocity. He used perturbation method to analysis the stability and density wave of traffic flow. In traffic flow the changing density wave from non-uniform to uniform distribution is described by the Burgers equation. YU and Zhou [24] in 2014 discussed the triangular shock wave determined by Burgers' equation in the stable region with reductive perturbation method. Yacob et. al [22] applied inviscid Burgers' equation to model traffic flow and solve one- way traffic flow by the method of linear system. In 2018 Hartono et. al [14] used multiple-scale method to solve the same problem. N Binatari [7] found that homotopy perturbation method is more effective to solve Burgers' equation with boundary condition in traffic flow problem.

6. Conclusions

Various situations in traffic flow can be best described with the help of Burgers' equation. We discussed the Burgers' equation along with its applications in traffic flow and focused on the recent development on the various forms of Burgers' equation.

References

[1] Abazari R, Borhanifar A (2010), Numerical study of the solution of the Burgers and coupled Burgers equations by a differential transformation method, *Computers and Mathematics with Applications*, **59**: 97-101.

[2] Adegboyegun B J (2013), An explicit solution of Burgers equation with special kinematic viscosity using decomposition technique, *Pure and Applied Mathematics Journal*, **2**: 134-139.

[3] Aw A, Klar A, Materne T, Rascle M (2002), Derivation of continuum traffic ow models from microscopic follow-the-leader models, *SIAM, Appl. Math*, **63**: 259-278.

[4] Aw A ,Rascle M (2000), Resurrection of 'second order' models of traffic flow, *SIAM, Appl. Math*, **60**: 916-938.

[5] Bateman H (1915), Some recent researches on the motion of fluids, *Monthly Weather Review*, **43**:163.

[6] Benton E R , Platzman G W (1972), A table of solutions of the one dimensional Burgers equation, *Quarterly of Applied Mathematics*, **30**: 195-212.

[7] Binatari N (2019), A comparison between Cole-Hopf Transformation and Homotopy Perturbation Method for Viscous Burger Equation in Traffic Flow, *Journal of Physics: Conference Series*, 1320.

[8] Bogaevskii I A (2004), Matter evolution in Burgulence, *Sbornik: Mathematics*, 1-12.

[9] Bonkile M P, Awasthi A, Lakshmi C, Mukundan V (2018), A systematic literature review of Burgers equation with recent advances, *Pramana - J Phys*, **90:** 69.

[10] Chowdhury D, Santen L, Schadschneider A (2000), Statistical physics of vehicular traffic and some related systems, *Physics Reports*, **329**: 199-329.

[11] Daganzo C (1995), Requiem for second-order fluid approximation to traffic flow, *Transport. Res*. **29:** 277-286.

[12] Engelberg S (1998), The Stability of the Shock Profiles of the Burgers' Equation, *Applied Mathematics Letters*, **11**: 97-101.

[13] Greenshields B D (1934), The photographic method of studying traffic behavior, *Highway Research Board, Denison University*.

[14] Hartono H, Binatari N and Saptaningtyas F Y (2018), Regular Perturbation of Inviscid Burger Equation in a Traffic Flow Problem, *Journal of Physics: Conference Series*,**1097**.

[15] Khalifa A K , Noor K I and Noor M A (2011), Some numerical methods for solving Burgers equation, *International Journal of Physical Sciences*, **6:** 1702-1710.

[16] Kurt A, enesiz Y, and Tasbozan O (2015), On the Solution of Burgers Equation with the New Fractional Derivative, *Open Physics*, **1:** 355-360.

[17] Lighthill M J, Whitham G B (1955), On kinematic waves II, A theory of traffic ow on long crowded roads, *Department of Mathematics, University of Manchester*.

[18] Ohwada T (2009), Cole Hopf Transformation as numerical tool for the Burgers equation, *Applied and Computational Mathematics*, **8:** 107-113.

[19] Sun Lu, Jafaripournimchahi A, Kornhauser A, Hu W (2020), A new higher-order viscous continuum traffic flow model considering driver memory in the era of autonomous and connected vehicles, *Physica A: Statistical Mechanics and its Applications, Elsevier*, vol. **547(C)**.

[20] Vikram D, Chakroborty P, Mittal S (2013), Exploring the behavior of LWR continuum models of traffic flow in presence of shock waves, *Procedia - Social and Behavioral Sciences* **104**: 412 - 421.

[21] Ucar Y, Yagmurlu N M and Tasbozan O (2017), Numerical Solutions of the Modified Burgers Equation by Finite Difference Methods, *Journal of Applied Mathematics, Statistics and Informatics*, **13:** 19-30.

[22] Yacob J, Zulkii M, Zainuddin N (2017), Rosly S, Mathematical Modelling of Burger's Equation Applied In Traffic Flow, *ResearchGate*.

[23] Yu X (2002), Analysis of the stability and density waves for traffic flow, *Chinese Physics Soc. and IOP Publishing Ltd.*, **11**: 1128-1134.

[24] Yu L and Zhou B (2014), The Burgers Equation for a New Continuum Model with Consideration of Driver’s Forecast Effect, *Journal of Applied Mathematics*.

[25] Zhang H M (1999), A mathematical theory of traffic hysteresis, *Transportation Research Part B: Methodological*, **33**: 1-23.