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# Radiation Coordinates of Florides - McCrea - Synge

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**Abstract:** In this work we construct the element of volume vector $d\sigma_r$ of a surface of constant retarded distance around the trajectory of a charged particle with arbitrary motion in a Riemannian space. This constitutes a generalization of the method pioneered by Synge [1] in special relativity. The technique employed is suggested by the 'radiation coordinates' $y^r$ introduced by Florides-McCrea-Synge [2, 3] in the study of gravitational radiation.

**Keywords**: Radiation Coordinates, Surfaces of Constant Retarded Distance

## 1. Introduction

Here, the Florides-McCrea-Synge coordinates [2, 3] are used for the electromagnetic radiation and are considerably adapted to this purpose because, for such coordinates, the curved space behaves like a "flat space" in some aspects. That is, the use of $y^r$ implies that what was learned in Minkowski space can be naturally translated to a Riemannian space. Our expression for the element of volume vector $d\sigma_r$, of a surface of constant retarded distance, agrees with that obtained by Villarroel [4] by means of the procedure that DeWitt-Brehme [5] use when constructing a surface with constant instantaneous distance. However, we think that our method is simpler and more powerful, because it turns immediate the results on radiation tensors deduced in [6]. We shall use the World Function $\Omega$ of Ruse [7] which allows having covariant expansions in a curved space. This function remained forgotten for a long time, and its present relevance may be seen in [5, 8-20].

## 2. Radiation Coordinates

We assume the Dedekind (1868) [21, 22]-Einstein summation convention for the addition of repeated indices, and that the metric locally takes the form, $(\eta_{ab}) = (1,1,1,-1)$ at any event. In order to construct the radiation coordinates $y^r$ [2] we need a timelike curve $C$ (which in this case will be the electron trajectory) with an orthonormal tetrad on it:

$$\lambda(a)_{i'}\lambda(b)^{i'} = \eta_{ab}, \quad \lambda(a)i'\lambda^{(a)}j' = g_{i'j'}, \quad \lambda(a)^{i'} = \lambda^{i'} \tag{1}$$

where, $\lambda^{i'} = \dfrac{dx^{i'}}{ds}$ is the unitary tangent vector to *C*, and $x^r$ is a totally arbitrary coordinate system with $ds^2 = g_{ij} \cdot dx^i \cdot dx^j$. The primed indices label points on *C*. Now let us see how $x^r$ gives new coordinates: We parameterize the null geodesic *P'P* in the form $x^r(v)$ with

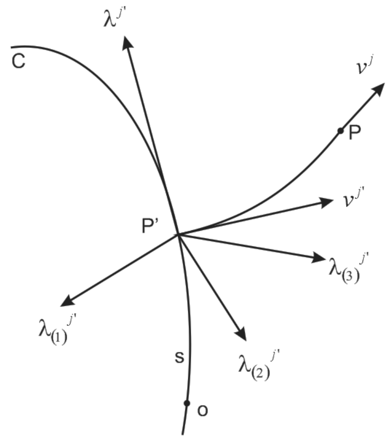


Fig. 1. For every *P* we construct the past sheet of its null cone which intersects to *C* in *P'* (retarded point associated to *P*).

$v = v_0$ at *P'*, and $v = v_1 > v_0$ at *P* with $V^r = \dfrac{dx^r}{dv}$ as its tangent vector, satisfying $V^r V_r = 0$. The assigned radiation coordinates to *P* are given by:

$$y^r = -\Omega_{,j'}\lambda^{(r)j'} + s\lambda_{j'}\lambda(r)j' \tag{2}$$

where $\Omega_{,j'}$ denote the covariant derivative of $\Omega$, see Synge [14]:

$$\Omega_{,j'} = -(v_1 - v_0)Vj', \quad \Omega_{,j'}\Omega^{j'} = 0, \tag{3}$$

so that $y^\sigma = -\Omega_{,j'}\lambda^{(\sigma)j'}, \quad y^4 = \Omega_{,j'}\lambda^{j'} + s$ which implies that in radiation coordinates the curve *C* is reduced to $y^{\sigma'} = 0$, $y^{4'} = s$. If we introduce the notation:

$$\xi_{j'} = -\Omega_{,j'}, \quad w = -\xi_{j'}\lambda^{j'} = \Omega_{,j'}\lambda^{j'} \tag{4}$$

then we obtain the form of the relation (9.3) of Synge [1] for flat space:

$$y^\sigma = y_\sigma = \xi_{j'}\lambda^{(\sigma)j'}, \quad y^4 = -y_4 = w + s\,, \tag{5}$$

in this sense the curved space behaves like a Minkowski space-time, which is very useful. On the other hand, at *P'* the metric tensor can be written in terms of the tetrad as:

$$g_{i'j'} = \lambda^{(\sigma)i'}\lambda_{(\sigma)j'} - \lambda_{i'}\lambda_{j'} \tag{6}$$

then $y^\sigma y_\sigma = \xi_{i'}\xi_{j'}\left(g^{i'j'} + \lambda^{i'}\lambda^{j'}\right) = w^2$ due to (3, 4), from where $\xi_{j'} = y^\sigma\lambda_{(\sigma)j'} + w\lambda_{j'}$, therefore $y^r$- $y^{r'}$ behaves like a null vector ($y^r$-$y^{r'}$)($y_r$ –$y_{r'}$)=0. Thus, our expressions are compatibles with (4, 5, 9) of [1]. Following the corresponding procedure in flat space let us introduce a new system of coordinates:

$$z^\sigma = y^\sigma, \quad z^4 = y^4 - \sqrt{y^\sigma y^\sigma} = s\,, \tag{7}$$

that is, $z^4$ remains constant on the null cone with vertex at $P'$. It is clear that the Jacobian of the transformation $\left(y^r \longrightarrow z^r\right)$ is equal to one, $J\,(z^i\,/\,y^r)$ = det $(\partial z^i\,/\,\partial y^r)$ = 1, therefore:

$$J\left(\frac{z^a}{x^b}\right) = J\left(\frac{y^a}{x^b}\right), \tag{8}$$

now let us calculate (8). We have that $\dfrac{\partial z^\sigma}{\partial x^i} = -\Omega_{i'i}\lambda^{(\sigma)i'} + N^\sigma \Omega_i, \quad \dfrac{\partial z^4}{\partial x^i} = -w^{-1}\Omega_i$ with

$N^\sigma = w^{-1}\left(\Omega_{i'j'}\lambda^{r'}\lambda^{(\sigma)j'} + \Omega_{r'}\dfrac{d}{ds}\lambda^{(\sigma)r'} + \Omega_{r'}\dfrac{d}{ds}\lambda^{(\sigma)r'}\right)$, where were employed the

properties $\dfrac{\partial x^{r'}}{\partial x^r} = \lambda^{r'} s_{,r} = -w^{-1}\lambda^{r'}\Omega_r, \quad \Omega_r = (v_1 - v_0)V_r$, hence:

$$J\left(\frac{z^a}{x^b}\right) = \varepsilon^{ijkm}\frac{\partial z^1}{\partial x^i}\frac{\partial z^2}{\partial x^j}\frac{\partial z^3}{\partial x^k}\frac{\partial z^4}{\partial x^m} = w^{-1}\varepsilon^{ijkm}\Omega_{j'i}\Omega_{r'j}\Omega_{t'k}\Omega_m\lambda^{(1)j'}\lambda^{(2)r'}\lambda^{(3)t'}, \tag{9}$$

for the skew-symmetric nature of the Levi-Civita density $\varepsilon^{ijkm}$. On the other hand, the World Function satisfies $\Omega_m = \Omega_{p'm}\Omega^{p'}$, substituting this into (9) we get:

$$J\left(\frac{z^a}{x^b}\right) = w^{-1}\det\left(-\Omega_{a'b}\right)\varepsilon_{j'r't'p'}\lambda^{(1)j'}\lambda^{(2)r'}\lambda^{(3)t'}\Omega^{p'}\;; \tag{10}$$

from (3) it is clear that $\Omega^{p'}$ can be written in terms of the tetrad:

$$\Omega^{p'} = a_\sigma\lambda^{(\sigma)p'} + a_4\lambda^{p'} \quad \therefore \quad w = \Omega_{p'}\lambda^{p'} = -a_4\,,$$

then, thanks to the skew-symmetry of $\varepsilon^{ijkm}$, equation (10) acquires the form:

$$J\left(\frac{z^a}{x^b}\right) = \det\left(-\Omega_{a'b}\right)\varepsilon_{j'r't'p'}\lambda^{(1)j'}\lambda^{(2)r'}\lambda^{(3)t'}\lambda^{(4)p'} = \det\left(-\Omega_{a'b}\right)\det\left(\lambda^{(r)i'}\right) = -g^{-1/2}(p')D\,, \tag{11}$$

where $D = -\left|-\Omega_{a'b}\right|, \quad g\left(P'\right) = -\left|g_{i'j'}\right|$. Let us introduce the notation:

$$\Delta = \overline{g}^{-1}D = g^{-1/2}\left(P\right)g^{-1/2}\left(P'\right)D, \quad g\left(P\right) = -\left|g_{ij}\right|, \tag{12}$$

thus from (11):

$$J\left(\frac{z^a}{x^b}\right) = -g^{1/2}\left(P\right)\Delta\,. \tag{13}$$

Taking into account the last identity it is clear the remark in [5] page 231 and [10] page 1251: the geodesics emerging from $P$ begin their intersection when $\Delta^{-1}=0$, arising the so-called ‘caustic surface’. We shall therefore accept that $P$ is near to $P'$, in order to have this only geodesic between them. The analysis performed allows consider the volume element of the curved space-time:

$$d^4x = \left| J\left(\frac{x}{z}\right) \right| d^4z = g^{-\frac{1}{2}}(P)\Delta^{-1} ds d^3 z, \tag{14}$$

but $z^{\sigma} = wp^{\sigma} = wp_i \lambda^{(\sigma)i'}$ with $p_i = w^{-1}\xi_{i'} - \lambda_{i'} =$ unitary spacelike vector :

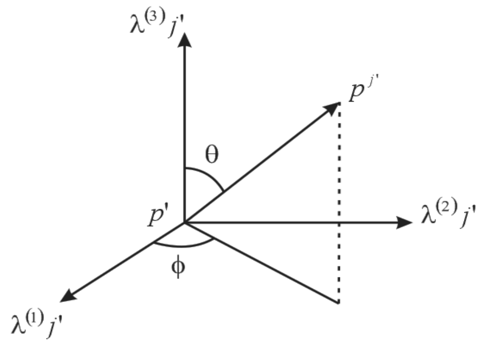


Fig. 2. The quantities $p^{\sigma}$ represent the components of $p^{j'}$ in the basis $\lambda^{(\sigma)j'}$

Therefore, $z^1 = w sin\theta \cos\phi$, $z^2 = w sin\theta sen\phi$, $z^3 = w\cos\theta$ which implies $d^3z = w^2 dw d\gamma$ where $d\gamma = sin\theta d\theta d\varphi$ is the element of solid angle in the rest frame of the charge. Then (14) adopts the form:

$$d^4x = g^{-\frac{1}{2}}(P)\Delta^{-1} w^2 ds dw d\gamma, \tag{15}$$

which together with (13) represents the generalization to Riemannian spaces of the results (9.15, 21) of Synge [1] (who made use of imaginary coordinates) for Minkowski space-time:

$$J\left(\frac{z^a}{x^b}\right) = -1, \quad d^4x = w^2 ds dw d\gamma. \tag{16}$$

In the next section we will apply (15) to the particular case of the surface $w$ = constant, which is important when studying the electromagnetic radiation

### 3. Surface of Constant Retarded Distance

Let us consider the 3-space $w$ = constant, then the covariant derivative $w_{;r}$ is orthogonal to that surface. It is therefore evident that its vector volume element is given by (where $d\sigma$ is the 3-element of volumen):

$$d\sigma_r = \left| w_{;a} w^{;a} \right|^{-\frac{1}{2}} w_{;r} d\sigma, \tag{17}$$

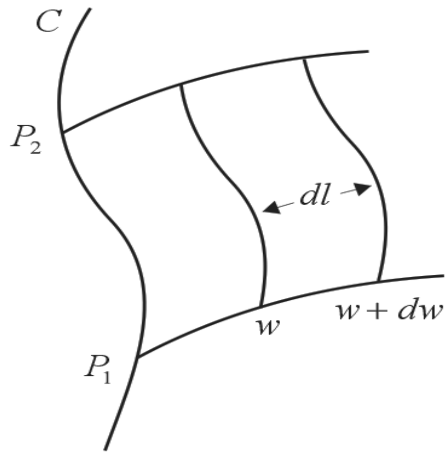


Fig. 3. Surface of constant retarded distance.

But when building the shell formed by $w,\, w + dw$ and the null cones at $P_1$ and $P_2$, we get for its 4-volume $d^4x = dld\sigma = \left|w_{;a}w^{;a}\right|^{-1/2} \cdot dw \cdot d\sigma$, and after comparison with (15) implies that $\left|w_{;a}w^{;a}\right|^{-1/2} d\sigma = g^{-1/2}(P)\cdot\Delta^{-1}w^2 dsd\gamma$, then (17) acquires the following form:

$$d\sigma_r = g^{-1/2}(P)\Delta^{-1}w^2 w_{;r} dsd\gamma. \tag{18}$$

On the other hand, from (4) we deduce the expression:

$$w_{;r} = \Omega_{i'r}\lambda^{i'} - w^{-1}\left(\Omega_{i'j'}\lambda^{i'}\lambda^{j'} + \Omega_{i'}\frac{d}{ds}\lambda^{i'}\right)\Omega_r = \hat{o}_r - w^{-1}(X+W)\Omega_r\,, \tag{19}$$

where we used the notation $\hat{o}_r = \Omega_{i'r}\lambda^{i'},\;\; X = \Omega_{i'j'}\lambda^{i'}\lambda^{j'}, W = \Omega_{i'}\frac{d}{ds}\lambda^{i'} = \Omega_{i'}\mu^{i'}.$

The substitution of (19) into (18) provides the result (3.35) of [4]:

$$d\sigma_r = g^{-1/2}(P)\Delta^{-1}w\left[w\hat{o}_r - (X+W)\Omega_r\right]dsd\gamma, \tag{20}$$

which is the generalization to curved spaces of the result (10.6) in [1]. The deduction of (20) was simple thanks to the radiation coordinates. Nevertheless, the usefulness of $z^r$ goes far beyond that; in our opinion, its true importance lies on the analogies that we can establish with the Minkowski space-time, which will be seen more clearly in the next section.

## 4. Radiation Tensors

In a flat space we have the following radiative part of the Maxwell tensor corresponding to the Liénard-Wiechert retarded field [23]:

$$\underset{R}{T_{rs}} = e'^2\, w^{-4}\left(\mu^2 - w^{-2}W^2\right)\xi_r\xi_s, \quad e' = \frac{e}{4\pi}, \tag{21}$$

with $\mu^2 = \mu_r \mu^r$, $\mu_r = \frac{d\lambda_r}{ds}$, $w = -\xi_r \mu^r$, $W = -\xi_r \lambda^r$, which satisfies:

$$\underset{R}{T_{rs}} \xi^s = 0, \tag{22}$$

$$\underset{R}{T_{rs}}{}^{,s} = 0. \tag{23}$$

A tensor field is said to be of the radiative type when it satisfies the properties (22) and (23). The continuity equation (23) is consequence of:

$$\left(\mu^2 w^{-4} \xi_r \xi_s\right)^{,s} = 0, \quad \left(w^{-6} W^2 \xi_r \xi_s\right)^{,s} = 0, \tag{24}$$

which in turn are particular cases of the identity:

$$\left[f\left(\mu^2\right) w^{-n} W^m \xi_r \xi_s\right]^{,s} = 0, \quad -n-m=-4, \tag{25}$$

$f$ being an arbitrary function of $\mu^2$. It seems natural to wonder whether (21) can be extended to the curved space. The answer is positive under the two following prescriptions:

a).- Identify $\xi_r$ with $-\Omega_r$, see (4).

b).- Multiply (21) by $\left| J\left(\frac{z^a}{x^b}\right) \right| = g^{\frac{1}{2}}(P)\Delta$ due to the fact that $d^4x$ contains the factor $g^{-\frac{1}{2}}(P)\Delta^{-1}$ with respect to the corresponding expression for the flat space, see (16).

Thus

$$\underset{R}{T_{rs}} = e'^2 g^{\frac{1}{2}}(P)\Delta w^{-4}\left(\mu^2 - w^{-2}W^2\right)\Omega_r \Omega_s \tag{26}$$

satisfies (23) with covariant derivative, due to the fact that the validity of (22) turns out to be evident. We can also expect the generalization of (24):

$$\left[g^{\frac{1}{2}}(P)\Delta\mu^2 W^{-4}\Omega_r\Omega_s\right]^{;s} = 0, \quad \left[g^{\frac{1}{2}}(P)\Delta w^{-6} W^2 \Omega_r \Omega_s\right]^{;s} = 0, \tag{27}$$

besides from (15) and (26) we have:

$$\underset{R}{T_{rs}} d^4x = e'^2 w^{-2}\left(\mu^2 - w^{-2}W^2\right)\xi_r \xi_s ds dw d\gamma \tag{28}$$

which is important when performing some integrations around the world line of the charged particle. It is worth noting that (26) and (27) correspond to the results (2.28,…,31) of Villarroel [6]. However, in our approach they can be obtained in a natural way by means of an explicit correspondence with the Minkowski space-time. The verification of (27) can be found in the work of the aforementioned author.

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