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A Short Note on Hyper-geometric Expression

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Abstract: In this short note, we exhibit an elementary deduction of the Borwein-Choi-Pigulla relation for $\log(1+z)+\sum_{k=1}^{n-1}\frac{(-z)^k}{k}$ in terms of the Gauss hyper-geometric function.

Keywords: Hyper-geometric function, Borwein-Choi-Pigulla's formula.

1. Introduction

In this paper, we briefly discuss on the hyper-geometric expression of $\log(1+z)+\sum_{k=1}^{n-1}\frac{(-z)^k}{k}$. The researchers Borwein-Choi-Pigulla [3] employed continued fractions to obtain the following identity [2]:

$$\log\ (1+z)+\sum_{k=1}^{n-1}\frac{(-z)^k}{k}=-\frac{(-z)^n}{n}\ {}_2F_1(n,1;\ n+1;-z),\qquad n=2,3,\dots \tag{1}$$

Here we show a simple deduction of (1) via the techniques of [4, 5] to convert a summation into a hyper-geometric function.

2. Borwein-Choi-Pigulla Formula

We have the known Taylor series for the logarithm function:

$$\log\ (1+z)-z=-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\cdots=-\frac{z^2}{2}\left(1-\frac{2z}{3}+\frac{2z^2}{4}-\frac{2z^3}{5}+\cdots\right), \tag{2}$$

$$=-\frac{z^2}{2}\sum_{k=0}^{\infty}u_k\ ,\qquad u_k=\frac{2(-1)^k z^k}{k+2},$$

and it is simple to apply the techniques of [4, 5] to convert a summation into a Gauss hyper-geometric function, in fact,

$u_0=1$ & $\frac{u_{k+1}}{u_k}=\frac{(k+2)\,(k+1)\,(-z)}{(k+3)\,(k+1)}$, therefore, we have

$$\log\,(1+z)-z=-\frac{z^2}{2}\ {}_2F_1(2,\ 1;\ 3;-z).$$

Similarly, from (2):

$$\log(1+z) - z + \frac{z^2}{2} = \frac{z^3}{3}\sum_{k=0}^{\infty}\frac{3(-1)^k z^k}{k+3}$$
$$= \frac{z^3}{3}\ {}_2F_1(3, \quad 1;\ 4; -z)$$

and in general:

$$\log(1+z) + \sum_{k=1}^{n-1}\frac{(-z)^k}{k} = -\frac{(-z)^n}{n}\sum_{k=0}^{\infty}\frac{n\,(-z)^k}{k+n}$$
$$= -\frac{(-z)^n}{n}\ {}_2F_1\,(n, 1;\ n+1;\ -z)$$

in harmony with the result (1) due to Borwein-Choi-Pigulla.

As always, a formula for *log* leads correspondingly to one for *arc tan* [2]:

$$\arctan z - \sum_{k=0}^{n-1}\frac{(-1)^k z^{2k+1}}{2k+1} = \frac{(-1)^n z^{2n+1}}{2n+1}\ {}_2F_1\left(n+\frac{1}{2},\ 1;\ n+\frac{3}{2}; -z^2\right). \tag{3}$$

3. Conclusion

Our procedure shows that the techniques of [4, 5] are very useful to translate a summation in terms of hyper-geometric functions, and thus to give simple proofs for several important formulae in the literature [1].

References

[1] Barrera FV, López-Bonilla J and López-Vázquez R (2016), On the Bailey-Borwein-Plouffe formula for π. *Prespacetime Journal* **7**(10) : 1408-1410.

[2] Borwein J, Bailey D and Girgensohn R (2004), *Experimentation in Mathematics. Computational paths to discovery.* A. K. Peters, Natick, Mass. USA.

[3] Borwein JM, Choi S and Pigulla W (2005), Continued fractions of tails of hypergeometric series. *Am. Math. Monthly*, Dec.

[4] Koepf W (1998), *Hypergeometric summation. An algorithmic approach to summation and special function identities.* Vieweg, Braunschweig/Wiesbaden.

[5] Petkovsek M, Wilf HS and Zeilberger D (1996), *A = B, symbolic summation algorithms.* A. K. Peters, Wellesley, Mass.