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A Note on Feasibility and Optimality of Transportation Problem

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Abstract: Transportation problem is one of the predominant areas of operations research, widely used as a decision making tool in engineering, business management and many other fields. In this paper, we present a brief literature review of transportation problem with its mathematical models in balanced and unbalanced cases. We report the basic feasible solution and hence the methods to attain optimal solution of the balanced transportation problem. Finally, we describe the primal-dual case of the problem with counter examples.

Keywords: Transportation, optimality, basic feasible solution, linear programming

1. Introduction

Transportation problem is one of the most interesting linear programming problems concerning with the distribution of products or services. The production and transportation have crucial role simultaneously to balance the overall supply chain of manufacturing companies. Whenever any product is produced by industry it has to reach to its end users. Consumers may lie far away from the industry. Therefore, transportation is essential to keep the end users in access of various goods and services produced. As the name suggests, transportation problem deals with logistic operations of various resources and finished goods from one place to another. The key factor is to decide the quantity, costs and the routes of transportation. Sources (factories) supply their goods to different destinations (warehouses) according to their demand. Every source aims to minimize the cost of transportation. This is how transportation problem occurs. Transportation model is useful for selecting the routes which will incur the minimum travelling cost that can also be used in making decision in facility locations as well. With the help of this model, we can determine the location of a new factory or manufacturing plant in such a way that its production cost can be minimized. The problem was formalized by French Mathematician G. Monge in 1781. A Russian mathematician L. V. Kantorovich made some major advancement in this field during and after the World War II to solve the post war problems. This is why the problem is sometimes stated as Monge-Kantorovich problem as well. Hitchcock (1941) gave the mathematical description of the transportation problem and Koopman stated the concept of optimum solution noticed on 1950 [1].

The two main objectives of transportation problem are to minimize the cost of shipping and maximize the profit of shipping from m sources to n destinations. This method is not only used to solve transportation problem but also deals with job assignment problem which aims at assigning various jobs to different employees so that maximum skill can be utilized with minimum cost. For example, suppose a biscuit company in Nepal has m factories located in m places which supplies the end products to n destinations. If the transportation costs from every source to all destinations is known, then transportation problem will determine the routes in such a way that the minimum possible transportation cost incurs.

There are two types of transportation problem, namely balanced transportation problem and unbalanced transportation problem. The balanced transportation problem is the one which has quantity of demand equal to that of the quantity to be supplied whereas in unbalanced transportation problem quantity of demand is not equal to the quantity of supply. The unbalanced transportation problem is converted into the balanced one by adding dummy row or dummy column. So it suffices to obtain the optimal solution of balanced transportation problem. The schematic view of transportation of goods from various sources to various destinations is given in Fig.1, where $S_1, S_2, \cdots, S_m$ denote the source nodes and $D_1, D_2, \cdots, D_n$ denote the destination nodes:

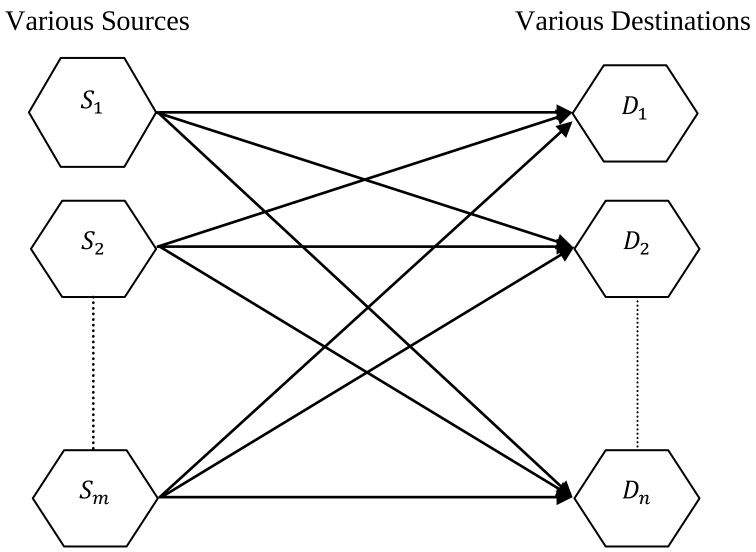

Figure 1: Transportation of goods from different sources to destinations

One of the special cases of transportation problem is transshipment problem which consider the real life situation. In transportation problem we assume sources as shipper of goods and markets as the receiver, but sometimes it is economical to supply goods from one source to another before reaching market or from one destination to other [3]. Let us assume that a company has two factories A_1 and A_2 situated at two different places and both produce equal amount of products but demand at A_1 is less than the production whereas it is high in case of A_2, so company will firstly dispatch its goods from A_1 to A_2. Also there may be many intermediate nodes between any two nodes in this problem.

The rest of the paper is planned as follows: the transportation problem is formulated in Section 2. The basic feasible solutions and hence the optimal solutions are discussed in Section 3. The over production case and under production case are briefly modeled in Section 4. The duality is shortly presented in Section 5 and the last Section concludes the paper.

2. Mathematical Formulation of Transportation Problem

Let X_{ij} = quantity of product supplied, C_{ij} = cost of transportation of unit quantity from i^{th} source to j^{th} destination, a_i = amount of quantity of the product available at i^{th} source and b_j

= amount required in j^{th} destination. The objective is to minimize the total cost of transportation which is formulated as:

$$\text{minimize} \qquad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij} \qquad \text{.........(1)}$$

$$\text{subject to} \qquad \sum_{j=1}^{n} X_{ij} = a_i\,, \quad i = 1, 2, \ldots\ldots, m \qquad \text{.........(2)}$$

$$\sum_{i=1}^{m} X_{ij} = b_j\,, \quad j = 1, 2, \ldots\ldots., n \qquad \text{.........(3)}$$

$$X_{ij} \geq 0, \quad \forall i, j \qquad \text{.........(4)}$$

A necessary and sufficient condition for the existence of a feasible solution for the transportation problem is that the total supply is equal to the total demand, *i.e.*, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. This implies that the problem is balanced transportation problem. Here, the expression (1) represents the objective function of the problem which indicates that the total cost of the transportation is to be minimized. The constraints (2) and (3) are availability and demand constraints respectively, jointly known as rim requirements. The negative quantity cannot be supplied from sources to destinations. This non-negativity is represented by the constraint (4). Transportation problem can be represented in tabular form as given below which is known as transportation table.

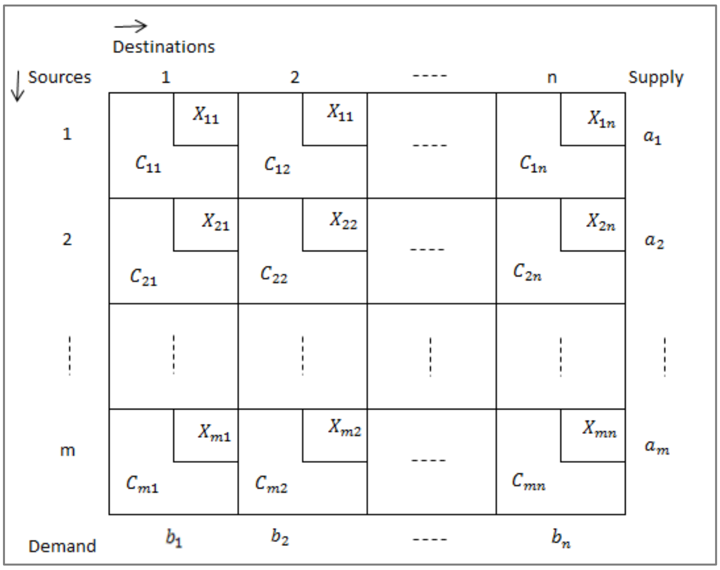

Figure 2: Transportation table with given demand and supply

3. Solution of Balanced Transportation Problem

The solution procedure of balanced transportation problem has two stages. The first is basic feasible solution and the second is optimal solution, which are briefly discussed below.

3.1 Basic Feasible Solution

Any solution $X_{ij} \geq 0$ is said to be a feasible solution of a transportation problem if it satisfies the constraints. The feasible solution is said to be basic feasible solution if the number of non-negative allocations is equal to $(m+n-1)$ while satisfying all rim requirements, *i.e.,* it must satisfy requirement and availability constraint [5]. There are three ways to get basic feasible solution:

A. North West Corner Rule

Hitchcock gave this simplest method to get the initial basic feasible solution. But it does not guarantee the best solution as cost factor is not considered. Salvemini in 1939 and Frechet in 1951 also described this method [1]. This method is described in the following steps:

Step I: Select the north-west corner cell of the table and allocate the maximum possible value which is minimum amongst the demand and supply, *i.e.*, $X_{11} = \min(a_1, b_1)$.

Step II: If $b_1 > a_1$, move down vertically to the second row and make the second allocation of magnitude $X_{21} = \min(a_2, b_1 - X_{11})$ in the second row first column. If $b_1 < a_1$, move towards right to the second column and make the second allocation $X_{12} = \min(a_1 - X_{11,} b_2)$ in the cell.

If $b_1 = a_1$, then arbitrarily move either towards next column or next row and allocate cell equal to zero.

Step III: Repeat steps 1 and 2 until all the supply and demand values are exhausted.

B. Minimum Cost Method or Matrix Minima Method

The solution derived from this method is better than North-West Corner Method but it may also take much iteration to reach the basic feasible solution [14, 6]. This method works as follows:

Step I: Select the cell with the minimum cost and allocating the maximum possible quantity, *i.e.*, minimum between supply and demand. If there is tie, then randomly allocate the values. Delete the row or column or both whichever is satisfied.

Step II: If $X_{ij} = a_i$, cross off the i^{th} row of the table and decrease b_j by a_i.

If $X_{ij} = b_j$, cross off the column and decrease a_i by b_j.

If $X_{ij} = a_i = b_j$, cross of any one either column or row.

Step III: Repeat above steps until all rim requirements are fulfilled.

C. *Vogel's Approximation Method or Regret Method*

This method is better than both the above methods because it goes to one more level of detailing by computing the penalty associated with non allocation at the minimum cost. In terms of computational requirement, Vogel take more time as this method gives superior starting basic feasible solution. It is expected to take less iteration to reach optimum solution. The Vogel's approximation method (VAM) usually produces an optimal or near- optimal starting solution. Generally, it is found that VAM yields an optimum solution in 80 percent of the problems [7]. The steps of this method are:

Step 1: Evaluate the difference between the lowest two cells in all rows and columns.

Step 2: Select the row or column with the largest difference. In case of tie choose arbitrarily.

Step 3: Allocate as much as possible to the lowest-cost cell in the selected row or column.

Step 4: Repeat above three steps until all rim requirements are satisfied.

3.2 Optimal Solution

A feasible solution of transportation problem is said to be optimal if it minimizes the total cost of transportation. There always exists an optimal solution to a balanced transportation problem [17]. We start with initial basic feasible solution to reach optimal solution which is obtained from above three methods. We then check whether the number of allocated cells is exactly equal to $m+n-1$, where m and n are number of rows and columns respectively. It works on the assumption that if the initial basic feasible solution is not basic, then there exists a loop. Here, we explain the two methods to attain the optimality.

A. *Stepping Stone Method*

The concept of stepping stone method (SSM) was introduced by Cooper and Charnes in 1945. This method involves formation of a loop which is a closed path traced by ordered set of cell that starts from an unoccupied cell but rest of the cells are occupied. Consecutive cells in loop lies either in same row or same column. The iterative procedure of stepping stone method is as follows: [8, 10, 13]

Step I: Select an unoccupied cell. Starting from the cell make a loop and assign (+) and (-) signs alternatively on each corner cell of the closed path just traced and begin with the plus sign at unoccupied cell to be evaluated.

Step II: Calculate an improvement index by first adding the unit cost figure with plus sign and subtracting the unit cost with minus sign.

Step III: Repeat the same above steps with all unoccupied cells.

Step IV: If any one of the improvement index appears to be negative, then it indicates that the given basic feasible solution is not optimal and implies that cost of transportation can be reduced further. Therefore, allocate the maximum possible value in the selected cell which is minimum of the value with negative sign.

Step V: Repeat above steps until all improvement indexes become positive or zero, which indicates optimum solution.

B. Modified Distribution Method or U-V Method

The modified distribution (MODI) method is the modified version of the stepping stone method (SSM). But MODI method is preferred over SSM as it overcomes the problem of evaluating all unoccupied cell which is encountered by SSM. Hence, it is convenient to use this method while solving large problems [9, 14]. The MODI method requires the inclusion of some sets of numbers $u_i's\ (u_1, u_2, u_3......u_m)$ and $v_j's\ (v_1, v_2, v_3......v_n)$ which helps to determine the penalty costs. The procedure to attain optimal solution of transportation problem using MODI method is as follows:

Step I: Assign u_i and v_j for $i = 1 - m$ and $j = 1 - n$ in the upper and right corner of the transportation table respectively. Now put $u_1 = 0$ and derive other values of u_i's and v_j's using formula $C_{ij} = u_i + v_j$

Step II: Determine the penalties, *i.e.*, $P_{ij} = u_i + v_j - C_{ij}$

Step III: If all penalty costs are positive or zero then it indicates optimal solution. If not, then allocate '$\boldsymbol{\theta}$' to the cell having most negative P.

Step IV: Starting from the selected cell make one closed loop and alternately assign + and - sign.

Step V: Look for the maximum possible increment in the value of '$\boldsymbol{\theta}$' which is minimum amongst the (-) sign unit cost.

Step VI: Now adding and subtracting C_{ij}'s by the value of '$\boldsymbol{\theta}$',where there is (+) and (-) sign respectively.

Step VII: Repeat above procedure until optimum solution is reached.

4. Unbalanced Transportation Problem

In real life situation, it is very rare to have balanced transportation problem so normally we deal with unbalanced transportation problem. This situation occurs when demand exceeds supply or when demand is less than the supplied goods. The two cases of the unbalanced transportation problem are formulated as below.

Case I: When demand > supply (Underproduction), then problem is modeled as follows:

$$\text{minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij} \qquad ....(5)$$

$$\text{subject to} \quad \sum_{j=1}^{n} X_{ij} = a_i\,, \quad i = 1,2,........,m \qquad ....(6)$$

$$\sum_{i=1}^{m} X_{ij} \le b_j\,, \quad j = 1,2,........,n \qquad ....(7)$$

$$\text{and} \quad X_{ij} \ge 0, \quad \forall i, j \qquad ....(8)$$

where the quantity of supply is less than the quantity demanded $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$. The problem of underproduction can be handled by including dummy supply or dummy row whose supply quantity will be $\sum_{i=1}^{m} b_j - \sum_{j=1}^{n} a_i$, but every C_{ij} of that row is equal to zero. Then the problem is converted into balanced transportation problem.

Case II: When supply > demand (Overproduction), then problem is modeled as follows::

minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij}$ ....(9)

subject to $\sum_{j=1}^{n} X_{ij} \leq a_i, \quad i = 1,2,.........,m$ ......(10)

$\sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1,2,........,n$ ......(11)

and $X_{ij} \geq 0, \quad \forall i, j$ ......(12)

where the quantity of supply is greater than the quantity demanded $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$. The problem of overproduction can be handled by including dummy destination or dummy column whose supply quantity will be $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ but every C_{ij} of that column is equal to zero. Again the problem of second type unbalanced problem is also converted into balanced transportation problem [4, 2, 11]. After converting unbalanced transportation problem into the balanced one, it can be obviously solved by using the methods discussed above. The transportation problem is solvable in polynomial time. Note that one special case of transportation problem, namely assignment formulation, is also very efficient to solve other related problems as well, for example the single-level just-in-time sequencing problem in mixed-model systems is reduced into assignment problem and solved efficiently [15].

5. Duality of Transportation Problem

Duality is the concept of analyzing a similar linear programming problem (LPP) with different aspect [12]. Every LPP can be transformed into another LPP problem, for example, profit maximization problem can be dealt as a problem of cost minimization. The original problem is called primal and transformed problem is known to be its dual. As dual of the dual is primal itself, it is immaterial to point out which problem is primal or dual. However, duality simplifies the complexities of the primal problem. There are certain rules to follow for the formation of dual problem:

- If the primal is maximization, then the dual problem is minimization and vice-versa.
- Number of constraints and number of variables of primal problem will be equal to number of variable and constraints respectively of the dual problem.
- The right hand side constants of primal problem appears in the objective function of dual whereas the constants in objective function gets converted to right hand side constants.
- The coefficient matrix of dual is the transpose of coefficient matrix of its primal.
- If the primal have constraints having less than or equal to type (≤), then its dual will have constraints with greater than or equal to type (≥) and vice-versa.
- If the r^{th} constraint of primal is equality type (=), then r^{th} variable of dual is unrestricted and vice-versa.
- The constraints and variables of primal correspond to variables and constraints of the dual respectively. Therefore, if primal consists of *m* constraints and n variables, its dual will have *n* constraints and *m* variables.

If the primal problem is difficult to solve via available methods, it is converted into its dual which significantly reduces the computational time and difficulty of the solution procedure as well. The following two examples of LPP are helpful to visualize the duality concept.

Example 1: The primal problem and its corresponding dual are given below respectively:

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3$

subject to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3$

and $x_1, x_2, x_3 \ge 0$

Minimize $F = b_1y_1 + b_2y_2 + b_3y_3$

Subject to $a_{11}y_1 + a_{21}y_2 + a_{31}y_3 \ge c_1$

$a_{12}y_1 + a_{22}y_2 + a_{32}y_3 \ge c_2$

$a_{13}y_1 + a_{23}y_2 + a_{33}y_3 \ge c_3$

and $y_1, y_2, y_3 \ge 0$

Example 2 (Diet Problem): Let us assume that minimum daily requirement of vitamin A and B is given and the content of two vitamins in food F1 and F2 with per unit cost of food is provided as follows:

Vitamin	Food		Minimum daily requirement
	F1	F2	
A	7	8	75
B	9	11	83
Cost (per unit)	10	15	

Now objective of the diet problem is to determine the amount of intake of food *F1* and *F2* in such a way that the cost incurred is minimum taking into consideration that minimum daily requirement is fulfilled. Let x_1 and x_2 be the number of units of food *F1* and *F2* to be purchased. This diet problem is formulated as follow:

minimize $Z = 10x_1 + 15x_2$

subject to $7x_1 + 8x_2 \geq 75$

$9x_1 + 11x_2 \geq 83$

and $x_1, x_2 \geq 0$

This LPP problem can be transformed to its dual. Assume that a dealer have to sell the foods *F1* and *F2* and his objective is to fix the per unit prices of foods in such a way that he receives the maximum price after selling foods without exceeding the per unit cost of foods. Let y_1 and y_2 be the per unit cost of foods *F1* and *F2*. Hence the dual of the diet problem can be formulated as

maximize $F = 75y_1 + 83y_2$

subject to $7y_1 + 9y_2 \leq 10$

$8y_1 + 11y_2 \leq 15$

and $y_1, y_2 \geq 0$

Now the transportation problem formulation $(1)-(4)$ is transferred into its dual as below:

maximize $\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \quad \forall i,\ j$ ......(13)

subject to $u_i + v_j \leq C_{ij}$ ......(14)

where u_i and v_j are unrestricted.

6. Conclusion

The globalization of business operations has created tough competition among organizations, and thus it has become necessary to attain optimal solution to every managerial and business problem. In this paper, we have discussed about some methods for attaining optimality in case of transportation problem. This problem is solved in two phases; firstly we get the initial basic feasible solution which includes three methods out of which Vogel's Approximation method gives us the best solution. After this we compute for optimal solution for which we use two methods stepping stone method and modified distribution method. These problems are so easy and general that even non-mathematicians can also understand. There are many LP software like IBM ILOG CPLEX OPTIMIZATION Studio which is normally referred to as CPLEX, LINDO, AIMMS available for computation of transportation problem. The transportation problem is efficiently solved and known as solvable in polynomial time. So this problem belongs to the complexity class *P*. If some or all of the variables are restricted to be integers, then the problem turns to be much harder falling in the field of combinatorial optimization. This integrality constraint makes most of the production planning problems fall into the complexity class *NP*. The integer programming problem is *NP*-hard in general [16]. Our future work will be focused in this domain of discrete optimization.

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