A SURVEY ON GRUSS-TYPE INEQUALITIES BY MEANS OF DIFFERENT GENERALIZED FRACTIONAL INTEGRALS

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Abstract

In this survey paper, we review a generalization of Gruss - type inequality by means of two different operators, one through K - fractional integral and other through Katugampola fractional integral. We state some related theorems, corollaries and their proofs. We study how results are interrelated?

Key Words: *Fractional Calculus, Gruss - inequality, Katugampola fractional integral, K-fractional integral.*

1. Introduction and preliminaries

Fractional calculus is the study of integrals and derivatives in case non- integer orders; which is a generalized form of classical integrals and derivatives. The creation of fractional calculus gave rise to several results and important theories in mathematics, physics, engineering and other fields of science.

It is well known that inequalities have potential applications in the technology, scientific studies, and analysis and numerous mathematical problems such as approximation theory, statistical analysis, and human social sciences. Presently, authors have provided the unique version of such inequalities, which may be beneficial in the investigation of diverse forms of integrodifferential and difference equations.

In 1935, G.Gruss proved the renowned integral inequality,

$$\left|\frac{1}{b-a}\int_a^b v\,(x)u(x)dx-\frac{1}{(b-a)^2}\int_a^b v\,(x)dx\int_a^b u\,(x)dx\right|\leq\frac{1}{4}(M-m)(P-p)$$

Where u,v are two integrable functions on [a, b], satisfying the conditions

$$m\leq v(x)\leq M, p\leq v(x)\leq P, x\in[a,b], m, M, p, P\in R \quad [3]$$

Gruss inequality (1) connects the integral of the product of two functions with the product of their integrals. It is extensively identified that cntinuous and discrete cases of Gruss-type variants play a considerable job in examining the qualitative conduct of differential and integral equations.

There are numerous approaches to acquiring a generalization of Gruss inequality using different fractional integral operators. A remarkably large number inequalities of type (1) involving the special fractional integral (such as the Liouville, Riemann-Liouville, Erdelyi-Kober, Katugampola, Hadamard and Weyl etc. types) have been investigated by many researchers. Here, in two parts we represent the generalization of Gruss-type inequality first part contains using generalized K-fractional integrals and second part using Katugampola fractional integral. Using these two operators the classical Gruss inequality (1) become a class of new Gruss-type inequalities, that generalize inequalities obtained using other fractional integrals [4].

Definition 1. *Let $\sigma_1, \sigma_2 \in (-\infty, \infty)$ such that $\sigma_1 < \sigma_2$ and $\Psi(\zeta)$ be an increasing and positive monotone function on $(\sigma_1, \sigma_2]$. Then the left-sided and right-sided generalized K-fractional integrals of a function $\mathfrak{p}$ with respect to Ψ of order $\rho, \kappa > 0$ are defined by*

$$\Im_{\Psi,\sigma_1^+}^{\rho,\kappa}\mathfrak{p}(\varrho) = \frac{1}{\kappa\Gamma_\kappa(\rho)}\int_{\sigma_1}^{\varrho} \Psi'(\zeta)(\Psi(\varrho) - \Psi(\zeta))^{\frac{\rho}{\kappa}-1}\mathfrak{p}(\zeta)\mathrm{d}\zeta, \qquad (2)$$

and

$$\Im_{\Psi,\sigma_2^-}^{\rho,\kappa}\mathfrak{p}(\varrho) = \frac{1}{\kappa\Gamma_\kappa(\rho)}\int_{\varrho}^{\sigma_2} \Psi'(\zeta)(\Psi(\zeta) - \Psi(\varrho))^{\frac{\rho}{\kappa}-1}\mathfrak{p}(\zeta)\mathrm{d}\zeta \qquad (3)$$

where Γ_κ is the κ- gamma function.

Using generalized K-fractional integral defined in above Gruss inequality (1) is generalized through following theorems.

Theorem 1. *Let $\kappa, \rho, \delta > 0$, $\mathfrak{p}$ be a positive function on $[0, \infty)$ and Ψ be an increasing g and positive function on $[0, \infty)$ such that $\Psi'(x)$ is continuous on $[0, \infty)$ with $\Psi = 0$. Suppose that there exist integrable functions Θ_1, Θ_2 on $[0, \infty)$ such that*

$$\Theta_1(\varrho) \le \Theta_2(\varrho)$$

for all $\varrho \in [0, \infty)$. Then we have

$$\Im_{\Psi,0^+}^{\delta,\kappa}\Theta_1(\varrho)\Im_{\Psi,0^+}^{\rho,\kappa}\mathfrak{p}(\varrho) + \Im_{\Psi,0^+}^{\rho,\kappa}\Theta_2(\varrho)\Im_{\Psi,0^+}^{\delta,\kappa}\mathfrak{p}(\varrho) \ge \Im_{\Psi,0^+}^{\rho,\kappa}\Theta_2(\varrho)\Im_{\Psi,0^+}^{\delta,\kappa}\Theta_1(\varrho) + \Im_{\Psi,0^+}^{\rho,\kappa}\mathfrak{p}(\varrho)\Im_{\Psi,0^+}^{\delta,\kappa}\mathfrak{p}(\varrho).$$

Theorem 2. *Let $\kappa, \varrho > 0$, $\mathfrak{p}, \mathfrak{s}, \Theta_1, \Theta_2, \varphi_1$ and φ_2 be six integrable functions defined on $[0, \infty)$, and Ψ be an increasing and positive function on $[0, \infty)$ such that $\Psi'(x)$ is continuous on $[0, \infty)$ and $\psi(0) = 0$.*

If conditions $\Theta_1(\varrho) \le \mathfrak{p}(\varrho) \le \Theta_2(\varrho)$ and $\varphi_1(\varrho) \le \mathfrak{s}(\varrho) \le \varphi_2(\varrho)$ are satisfied, then one has

$$\left|\frac{\Psi^{\frac{\rho}{\kappa}}}{\Gamma_\kappa(\rho+\kappa)}\Im_{\Psi,0^+}^{\rho,\kappa}(\mathfrak{p}(\varrho)\mathfrak{s}(\varrho)) - \Im_{\Psi,0^+}^{\rho,\kappa}\mathfrak{p}(\varrho)\Im_{\Psi,0^+}^{\rho,\kappa}\mathfrak{s}(\varrho)\right| \le \sqrt{\mathcal{J}(\mathfrak{p}, \Theta_1, \Theta_2)\mathcal{J}(\mathfrak{s}, \varphi_1, \varphi_2)} \quad (4)$$

where function $\mathcal{J}(\mathfrak{p}, \Theta_1, \Theta_2)$ and $\mathcal{J}(\mathfrak{s}, \varphi_1, \varphi_2)$ are defined as in [1]

The proofs of theorems (1) and (2) can be seen in [1]

Corollary 3. *Let $m, M, n, N \in \mathbb{R}$, $\mathcal{J}(\mathfrak{p}, \Theta_1, \Theta_2) = \mathcal{J}(\mathfrak{p}, m, M)$ and $\mathcal{J}(\mathfrak{s}, \varphi_1, \varphi_2) = \mathcal{J}(\mathfrak{s}, n, N)$. Then inequality (4) reduces to*

$$\left|\frac{\Psi^{\frac{\rho}{\kappa}}(\varrho)}{\Gamma_\kappa(\rho+\kappa)}\Im_{\Psi,0^+}^{\rho,\kappa}(\mathfrak{p}(\varrho)\mathfrak{s}(\varrho)) - \Im_{\Psi,0^+}^{\rho,\kappa}\mathfrak{p}(\varrho)\Im_{\Psi,0^+}^{\rho,\kappa}\mathfrak{s}(\varrho)\right| \le \left(\frac{\Psi^{\frac{\rho}{\kappa}}(\varrho)}{2\Gamma_\kappa(\rho+\kappa)}\right)^2(M-m)(N-n).$$

Corollary 4. *Let $m, M, n, N \in \mathbb{R}$, $\Psi(\varrho) = \varrho$, $\mathcal{J}(\mathfrak{p}, \Theta_1, \Theta_2) = \mathcal{J}(\mathfrak{p}, m, M)$ and $\mathcal{J}(\mathfrak{s}, \varphi_1, \varphi_2) = \mathcal{J}(\mathfrak{s}, n, N)$. Then inequality (4) leads to*

$$\left|\frac{\varrho^{\frac{\rho}{\kappa}}}{\Gamma_\kappa(\rho+\kappa)}\Im^{\rho,\kappa}(\mathfrak{p}(\varrho)\mathfrak{s}(\varrho)) - \Im^{\rho,\kappa}\mathfrak{p}(\varrho)\Im^{\rho,\kappa}\mathfrak{s}(\varrho)\right| \le \left(\frac{\varrho^{\frac{\rho}{\kappa}}}{\Gamma_\kappa(\rho+\kappa)}\right)^2(M-m)(N-n).$$

Definition 2. *The K- fractional integral of the Riemann- Liouville type is defined as*

$$ {}_kJ_a^{\alpha}[f(t)] = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\mathrm{d}t; \alpha > 0, x > a. \qquad (5) $$

where f is a continuous function on [a, b]. The generalized gamma function

$$ \Gamma_k(\alpha) = \lim_{n\to\infty} \frac{n!\,k^n(nk)^{\frac{x}{k}-1}}{(x)_{(n,k)}}, k > 0; $$

with $(x)_{n,k}$ *isthe Pocchammer k - symbol for factorial function. Note that when* $k \to 1$, *then equation (5) reduces to the classical Riemann-Liouville fractional integral.*

Using operator ${}_kJ_a^{\alpha}[f(t)]$ defined by (5), the generalized Gruss-type integral inequality is given by following theorem.

Theorem 5. *Let f and g be two integrable function on [a, b] with* $\varphi < f(t) < \Phi$, $\psi < g(t) < \Psi$ *and let p be a positive function on [a, b]. Then for all t >0, k >0,* $\alpha > 0$, *we have:*

$$ |({}_kJ_a^{\alpha}[p(t)])({}_kJ_a^{\alpha}[pfg(t)]) - ({}_kJ_a^{\alpha}[pf(t)])({}_kJ_a^{\alpha}[pg(t)])| \le \frac{({}_kJ_a^{\alpha}[p(t)])^2}{4}(\Phi-\varphi)(\Psi-\psi). $$

Proof. Let us define the quantity

$$ H(\tau,\rho) = (f(\tau)-f(\rho))(g(\tau)-g(\rho)); \tau,\rho \in (a,t), a < t \le b. $$

multiplying (7) by $\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}P(\tau)$; *k>0,* $\tau \in (a,t)$, *and integrating the resulting identity with respect to* τ *from a to t, we can state that*

$$ \frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-\tau)^{\frac{\alpha}{k}-1} p(\tau)H(\tau,\rho)\mathrm{d}\tau = $$
$$ ({}_kJ_a^{\alpha}[pfg(t)]) - g(\rho)({}_kJ_a^{\alpha}[pf(t)] - f(\rho)({}_kJ_a^{\alpha}[pg(t)]) + f(\rho)g(\rho)({}_kJ_a^{\alpha}[p(t)]). $$

And then,

(5.1)

$$ \frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1} p(\tau)p(\rho)H(\tau,\rho)\mathrm{d}\tau\mathrm{d}\rho = $$
$$ 2({}_kJ^{\alpha}[p(t)])({}_kJ^{\alpha}[pfg(t)]) - 2({}_kJ^{\alpha}[pf(t)])({}_kJ^{\alpha}[pg(t)]). $$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we have: (5.2)

$$ (\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1} p(\tau)p(\rho)H(\tau,\rho)\mathrm{d}\tau\mathrm{d}\rho)^2 \le \frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1} $$
$$ p(\tau)p(\rho)(f(\tau)-f(p))^2\mathrm{d}\tau\mathrm{d}\rho \times \frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1} p(\tau)p(\rho)(g(\tau)-g(p))^2\mathrm{d}\tau\mathrm{d}\rho. $$

The right-hand side of (5.2) can be expressed as follows

(5.3) $$\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)(f(\tau)-f(p))^2\mathrm{d}\tau\mathrm{d}\rho = 2({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pf^2(t)])-2({}_kJ_a^\alpha[pf(t)])^2.$$

and

(5.4) $$\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1}p(\tau)p(\rho)(g(\tau)-g(p))^2\mathrm{d}\tau\mathrm{d}\rho = 2({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pg^2(t)])-2({}_kJ_a^\alpha[pg(t)])^2.$$

Now using (5.1), (5.3),and (5.4), we can write (5.2) as follows

(5.5) $$)({}_kJ_a^\alpha[pfg(t)])-({}_kJ_a^\alpha[pf(t)])({}_kJ_a^\alpha[pg(t)])]^2 \le [({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pf^2(t)])-({}_kJ_a^\alpha[pf(t)])^2]\times[({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pg^2(t)])-({}_kJ_a^\alpha[pg(t)])^2].$$

If we apply lemma (1) of with u = f, then u = g for (5.2), we obtain respectively

$$({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pf^2(t)])-({}_kJ_a^\alpha[pf(t)])^2 = [\Phi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pf(t)])][({}_kJ_a^\alpha[pf(t)])-\phi({}_kJ_a^\alpha[p(t)])] -({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[\Phi-f(t))(f(t)-\phi)p(t))]$$

and

$$({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pg^2(t)])-({}_kJ_a^\alpha[pg(t)])^2 = [\Psi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pg(t)])][({}_kJ_a^\alpha[pg(t)])-\psi({}_kJ_a^\alpha[p(t)])] -({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[\Psi-g(t))(g(t)-\psi)p(t))]$$

On the other hand, since

$$-({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[\Phi-f(t))(f(t)-\phi)p(t)]) \le 0$$

and

$$-({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[\Psi-g(t))(g(t)-\psi)p(t)]) \le 0$$

then, we have the inequalities respectively,

(5.6) $$({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pf^2(t)])-({}_kJ_a^\alpha[pf(t)])^2 \le [\Phi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pf(t)])][({}_kJ_a^\alpha[pf(t)])-\phi({}_kJ_a^\alpha[p(t)])]$$

and

(5.7)

$$({}_kJ_a^\alpha[p(t)])({}_kJ_a^\alpha[pg^2(t)])-({}_kJ_a^\alpha[pg(t)])^2 \le [\Psi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pg(t)])][({}_kJ_a^\alpha[pg(t)])-\psi({}_kJ_a^\alpha[p(t)])]$$

By taking into account (5.5), (5.6) and (5.7), we get the following inequality

(5.8)

$$)({}_kJ_a^\alpha[pfg(t)])-({}_kJ_a^\alpha[pf(t)])({}_kJ_a^\alpha[pg(t)])]^2 \le [\Phi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pf(t)])] [({}_kJ_a^\alpha[pf(t)])-\phi({}_kJ_a^\alpha[p(t)])]\times[\Psi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pg(t)])][({}_kJ_a^\alpha[pg(t)])-\psi({}_kJ_a^\alpha[p(t)])]$$

Since $4rs \le (r+s)^2, r,s \in \mathbb{R}$, *then it yields*

(5.9) $$4[\Phi({}_kJ_a^\alpha[p(t)])-({}_kJ_a^\alpha[pf(t)])][({}_kJ_a^\alpha[pf(t)])-\phi({}_kJ_a^\alpha[p(t)])] \le ((\Phi-\phi)({}_kJ_a^\alpha[p(t)])^2$$

And

(5.10)

$$4[\Psi({}_kJ_a^{\alpha}[p(t)]) - ({}_kJ_a^{\alpha}[pg(t)])][({}_kJ_a^{\alpha}[pg(t)]) - \psi({}_kJ_a^{\alpha}[p(t)])] \leq ((\Psi - \psi)({}_kJ_a^{\alpha}[p(t)])^2.$$

If (5.8)- (5.10) are taken into account together, then we get the inequality (6).

If $\alpha = k = 1$ *and p(t) = 1 then we get classical Gruss inequality on [a, t].*

2. Generalization using Katugampola fractional integral :-

Definition 3. [3] *Consider the space* $X_c^p(a,b), (c \in \mathbb{R}, 1 \leq P \leq \infty)$, *of those complex valued Lebesgue measurable function V on (a, b) for which norm* $\| V \|_{x_c^p} < \infty$, *such that*

$$\| V \|_{x_c^p} = (\int_x^b | x^c V|^p \frac{\mathrm{d}x}{x})^{\frac{1}{p}}, \ for 1 \leq p < \infty$$

and

$$\| V \|_{x_c^\infty} = supess_{x\in(a,b)}[x^c|V|]$$

In particular, when $c = \frac{1}{p}$; *the space* $x_c^p[a,b)$ *coincides with the space* $L_{(a,b)}^p$.

Definition 4. *[4] The left- and right- sided fractional integrals of a function V where* $V \in x_c^p(a,b)$, $\alpha > 0$ *and* $\beta, \rho, \eta, k \in \mathbb{R}$ *are defined respectively by*

$$^{\rho}\mathfrak{J}_{a^+;\eta,k}^{\alpha,\beta} V(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^p - \tau^p)^{1-\alpha}} V(\tau)\mathrm{d}\tau, \ 0 \leq a < x < b \leq \infty \ (8)$$

And

$$^{\rho}\mathfrak{J}_{b^-;\eta,k}^{\alpha,\beta} V(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_x^b \frac{\tau^{k+\rho-1}}{(\tau^p - x^p)^{1-\alpha}} V(\tau)\mathrm{d}\tau; \ 0 \leq a < x < b \leq \infty$$

if the integral exists. If we consider a = 0 in (8) then

$$^{\rho}I_{\eta,k}^{\alpha,\beta} V(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^p - \tau^p)^{1-\alpha}} V(\tau)\mathrm{d}\tau.$$

Definition 5. *Let* $x > 0, \alpha > 0, \ \rho, k, \beta, \eta \in \mathbb{R}$. *Then we define a function*

$$\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \rho^{-\beta} x^{k+\rho(\eta+\alpha)}.$$

If we take $\beta = \alpha$, *k = 0,* $\eta = 0$ *then definition (4) reduces to Katugampola fractional integral.*

In order to proof main theorem , which generalizes the inequality of gruss-Type, we state following two lemmas;

Lemma 6. *Let f and g be two integrable functions on* $[0,\infty)$. *Then for all* $\beta, k \in \mathbb{R}$, *x>0,* $\alpha > 0, \rho > 0, \eta \geq 0$ *and* $\gamma > 0$ *we have*

$$(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,k}^{\gamma,\beta} fg(x) + \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,k}^{\alpha,\beta} fg(x) - {}^{\rho}I_{\eta,k}^{\alpha,\beta} f(x) {}^{\rho}I_{\eta,k}^{\gamma,\beta} g(x) - {}^{\rho}I_{\eta,k}^{\gamma,\beta} f(x)\ {}^{\rho}I_{\eta,k}^{\alpha,\beta} g(x))^2$$
$$\leq (\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) {}^{\rho}I_{\eta,k}^{\gamma,\beta} f^2(x) + \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta) {}^{\rho}I_{\eta,k}^{\alpha,\beta} f^2(x) - 2\ {}^{\rho}I_{\eta,k}^{\alpha,\beta} f(x) {}^{\rho}I_{\eta,k}^{\gamma,\beta} f(x)) \times (\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) {}^{\rho}I_{\eta,k}^{\gamma,\beta} g^2(x)$$
$$+\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) {}^{\rho}I_{\eta,k}^{\alpha,\beta} g^2(x) - 2\ {}^{\rho}I_{\eta,k}^{\alpha,\beta} g(x)\ {}^{\rho}I_{\eta,k}^{\gamma,\beta} g(x))$$

Lemma 7. *Let u be an integrable function on* $[0,\infty)$ *satisfying* $m \le f(x) \le M$ *and* $p \le g(x) \le P$ *for* $m, M, p, P \in \mathbb{R}$ *and* $x \in [0,\infty)$. *Then for all* $\beta, k \in \mathbb{R}$, *x >0,* $\alpha > 0$, $\rho > 0, \eta > 0$ *and* $\gamma > 0$ *we have*

$$\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}u^2(x) + \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}u^2(x) - 2\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}u(x)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}u(x) = (M\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \overset{\rho}{-} I_{\eta,k}^{\alpha,\beta}u(x)) \times ({}^{\rho}I_{\eta,k}^{\gamma,\beta}u(x) - m\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)) + (M\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta) \overset{\rho}{-} I_{\eta,k}^{\gamma,\beta}u(x)) \times ({}^{\rho}I_{\eta,k}^{\alpha,\beta}u(x) - m\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)) - \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}(M-u(x))(u(x)-m) - \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}(M-u(x))(u(x)-m).$$

Theorem 8. *Let f and g be two integrable function on* $[0,\infty)$ *such that* $m \le f(x) \le M$ *and* $p \le g(x) \le P$ *for* $m, M, p, P \in \mathbb{R}$ *and* $x \in [0,\infty)$. *Then for all* $\beta, k \in \mathbb{R}$, *x >0,* $\alpha > 0$, $\rho > 0, \eta > 0$ *and* $\gamma > 0$ *we have*

$$(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}fg(x) + \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}fg(x) - {}^{\rho}I_{\eta,k}^{\alpha,\beta}f(x){}^{\rho}I_{\eta,k}^{\gamma,\beta}g(x) - {}^{\rho}I_{\eta,k}^{\gamma,\beta}f(x)\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}g(x))^2 \le [(M\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \overset{\rho}{-} I_{\eta,k}^{\alpha,\beta}f(x))({}^{\rho}I_{\eta,k}^{\gamma,\beta}f(x) - m\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)) + ({}^{\rho}I_{\eta,k}^{\alpha,\beta}f(x) - m\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)) (M\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta) \overset{\rho}{-} I_{\eta,k}^{\gamma,\beta}f(x))] \times [(P\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \overset{\rho}{-} I_{\eta,k}^{\alpha,\beta}g(x))({}^{\rho}I_{\eta,k}^{\gamma,\beta}g(x) - p\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)) + ({}^{\rho}I_{\eta,k}^{\alpha,\beta}g(x) - p\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)(P\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta) - {}^{\rho}I_{\eta,k}^{\gamma,\beta}g(x)].$$

Proof. Since $(M - f(x))(f(x) - m) \ge 0$, $(P - g(x))(g(x) - p) \ge 0$, x>0 and $\rho > 0$ we can write

$$-\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}(M-f(x))(f(x)-m) - \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}(M-f(x))(f(x)-m) \le 0 \qquad (9)$$

and

$$-\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}(P-g(x))(g(x)-p) - \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}(P-g(x))(g(x)-p) \le 0 \qquad (10)$$

Applying lemma (7) for f and g and using equation (9) and (10), we obtain

$$(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}f^2(x) + \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}f^2(x) - 2\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}f(x)) \le (M\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \overset{\rho}{-} I_{\eta,k}^{\alpha,\beta}f(x)) \times ({}^{\rho}I_{\eta,k}^{\gamma,\beta}f(x) - m\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)) + ({}^{\rho}I_{\eta,k}^{\alpha,\beta}f(x)) - m\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta))(M\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta) \overset{\rho}{-} I_{\eta,k}^{\gamma,\beta}f(x)).$$

and

$$(\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}g^2(x) + \Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}g^2(x) - 2\, {}^{\rho}I_{\eta,k}^{\gamma,\beta}\, {}^{\rho}I_{\eta,k}^{\alpha,\beta}g(x)) \le (P\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \overset{\rho}{-} I_{\eta,k}^{\alpha,\beta}g(x)) \times ({}^{\rho}I_{\eta,k}^{\gamma,\beta}g(x) - p\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta)) + ({}^{\rho}I_{\eta,k}^{\alpha,\beta}g(x) - p\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)) \times (P\Lambda_{x,k}^{\rho,\beta}(\gamma,\eta) \overset{\rho}{-} I_{\eta,k}^{\gamma,\beta}g(x)).$$

Taking the product of above two inequalities and using lemma (6), proof of theorem follows immediately.

Remark: - Putting $\alpha = \gamma = 1$, $\eta = 0$, k = 0 and $\rho \to 1$ in this theorem we get classical Gruss inequality. So, this is the theorem that generalized Gruss-Type integral inequality using Katugampola fractional integral.

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