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PROPERTIES RELATED TO WEIGHT FUNCTION

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Abstract

In this paper, we begin with brief discussion of theory of weights and A_p weight functions. We then state and prove some of the properties of A_P weight function using elementary analysis tools.

Key words: *A_1 weight function, Maximal functions, A_p weight function, Holder's Inequality.*

1. Introduction

The theory of weight are useful in boundary value problems for Laplace's equation and are much needed in extrapolation theory, vector-valued inequalities and estimates for certain class of non linear differential equation. Muckenhoupt (1970) characterized positive functions w for which the Hardy-Littlewood maximal operator M maps $L_p(R^n, w(x)dx)$ to itself. Muckenhoupt's characterization actually gave the better understanding of theory of weighted inequalities which then led to the introduction of A_p class and consequently the development of weighted inequalities [4]. To prove the results, some definition and results are in order:

Definition: The uncentered Hardy-Littlewood maximal operators on $\boldsymbol{R}^n$ over balls B is defined as

$$M(f)(x) = \sup_{x\in B} \operatorname*{Avg}_{B} |f| = \sup_{x\in B} \frac{1}{|B|}\int_B |f(y)|dy.$$

Similarly the uncentered Hardy-Littlewood maximal operators on R^n over cubes Q is defined as

$$M_c(f)(x) = \sup_{x\in Q} \operatorname*{Avg}_{Q} |f| = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

In each of the definition above, the suprema are taken over all balls B and cubes Q containing the point x. H-L maximal functions are widely used in Harmonic Analysis. For the details about the H-L maximal operators, see [1].

Definition: A locally integrable function on $\boldsymbol{R}^n$ that takes values in the interval $(0,\infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero.

Definition: A function $w(x)>0$ is called an A_1 weight if there is a constant $C_1>0$ such that

$$M(w)(x) \leq C_1\, w(x)$$

where M(w) is uncentered Hardy-Littlewood Maximal function given by

$$M(w)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B w(t)dt.$$

If w is an A_1 weight, then the quantity (which is finite) given by

$$[w]_{A_1} = \sup_{Q\ cubes\ in \boldsymbol{R}^n} \left(\frac{1}{|Q|} \int_Q |w(t)|dt \right) ||w^{-1}||_{L^\infty(Q)}$$

is called the A_1 characteristic constant of w.

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ is finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\ cubes\ in\ \boldsymbol{R}^n} \left(\frac{1}{|Q|} \int_Q |w(x)|dx \right) \left(\frac{1}{|Q|} \int_Q |w(x)|^{\frac{-1}{p-1}} dx \right)^{p-1}.$$

We note that in the above definition of A_1 and A_p one can also use set of all balls in $\boldsymbol{R}^n$ instead of all cubes in $\boldsymbol{R}^n$.

Finally, we state and prove some of the properties of weight functions:

Property 1: Let w_1 and w_2 be two A_1 weights and let $1 < p < \infty$. Then $w_1 {w_2}^{1-p}$ is an A_p weight and $[w_1 {w_2}^{1-p}]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1}$.

Proof: For every cube Q in $\boldsymbol{R}^n$, we have

$$\left(\frac{1}{|Q|} \int_Q w_1 {w_2}^{1-p} dx \right) \left(\frac{1}{|Q|} \int_Q (w_1 {w_2}^{1-p})^{\frac{-1}{p-1}} dx \right)^{p-1}$$

$$\leq \left(\frac{1}{|Q|} \int_Q w_1 \left\| w_2^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})}^{p-1} dx \right) \left(\frac{1}{|Q|} \int_Q \left\| w_1^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})}^{1/(p-1)} w_2\, dx \right)^{p-1}$$

$$= \left\| w_2^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})}^{p-1} \left(\frac{1}{|Q|} \int_Q w_1\, dx \right) \left\| w_1^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})} \left(\frac{1}{|Q|} \int_Q w_2\, dx \right)^{p-1}$$

Taking supremum on both sides, we have

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q w_1 {w_2}^{1-p} dx \right) \left(\frac{1}{|Q|} \int_Q (w_1 {w_2}^{1-p})^{\frac{-1}{p-1}} dx \right)^{p-1} \right]$$

$$\leq \sup_Q \left[\left\| w_2^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})}^{p-1} \left(\frac{1}{|Q|} \int_Q w_1\, dx \right) \left\| w_1^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})} \left(\frac{1}{|Q|} \int_Q w_2\, dx \right)^{p-1} \right]$$

$$\leq \sup_Q \left[\left(\frac{1}{|Q|} \int_Q w_1\, dx \right) \right] \left\| w_1^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})} . \sup_Q \left[\left(\frac{1}{|Q|} \int_Q w_2\, dx \right)^{p-1} \right] \left\| w_2^{-1} \right\|_{\mathrm{L}\infty(\mathrm{Q})}^{p-1}$$

This shows that $[w_1 {w_2}^{1-p}]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1}$. Consequently, we have $w_1 {w_2}^{1-p} \in A_p$.

Next we state and prove another property:

Property 2: Let $w_0 \in A_{p_0}$ and $w_1 \in A_{p_1}$ for some $1 \leq p_0, p_1 < \infty$. Let $0 \leq \theta \leq 1$ and define $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $w^{\frac{1}{p}} = w_0^{\frac{1-\theta}{p_0}} w_1^{\frac{\theta}{p_1}}$. Then w is in A_p showing that $[w]_{A_p} \leq [w_0]_{A_{p_0}}^{(1-\theta)\frac{p}{p_0}} [w_1]_{A_{p_1}}^{\frac{\theta p}{p_1}}$.

Proof: Let p', p_0' and p_1' be the conjugate exponent of p, p_0 and p_1 respectively. We have,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} - - - - - - - - - - - - - -(1)$$

This gives

$$1 - \frac{1}{p'} = (1-\theta)\left(1 - \frac{1}{p_0{'}}\right) + \theta\left(1 - \frac{1}{p_1{'}}\right).$$

Simplifying the above relation we get

$$\frac{1}{p'} = \frac{1-\theta}{p_0{'}} + \frac{\theta}{p_1{'}} - - - - - - - - - - - - - - -(2)$$

From (1) and (2) we immediately get,

$$1 = \frac{1}{\frac{p_0}{1-\theta}} + \frac{1}{\frac{p_1}{\theta}} \text{ and } 1 = \frac{1}{\frac{p_0{'}}{1-\theta}} + \frac{1}{\frac{p_1{'}}{\theta}}.$$

Given a cube Q in $\boldsymbol{R}^n$, we apply Holder's inequality with the exponent $\frac{p_0}{1-\theta}$ and $\frac{p_1}{\theta}$ to obtain

$$\frac{1}{|Q|}\int_Q w = \frac{1}{|Q|}\int_Q w_0^{\frac{(1-\theta)p}{p_0}} w_1^{\frac{\theta p}{p_1}}$$

$$\leq \left(\frac{1}{|Q|}\int_Q w_0\right)^{\frac{(1-\theta)p}{p_0}} \left(\frac{1}{|Q|}\int_Q w_1\right)^{\frac{\theta p}{p_1}}.$$

These yields

$$\frac{1}{|Q|}\int_Q w \leq \left(\frac{1}{|Q|}\int_Q w_0\right)^{\frac{(1-\theta)p}{p_0}} \left(\frac{1}{|Q|}\int_Q w_1\right)^{\frac{\theta p}{p_1}} - - - - - - - - - - - -(3)$$

We again apply Holder's inequality with the exponent $\frac{p_0{'}}{1-\theta}$ and $\frac{p_1{'}}{\theta}$ to obtain

$$\left(\frac{1}{|Q|}\int_Q w^{-\frac{p'}{p}}\right)^{\frac{p}{p'}} = \left(\frac{1}{|Q|}\int_Q w_0^{\frac{-(1-\theta)p'}{p_0}} w_1^{\frac{-\theta p'}{p_1}}\right)^{\frac{p}{p'}}$$

$$\leq \left[\left(\frac{1}{|Q|}\int_Q w_0^{\frac{-p_0{'}}{p_0}}\right)^{\frac{(1-\theta)p'}{p_0}} \left(\frac{1}{|Q|}\int_Q w_1^{\frac{-p_1{'}}{p_1}}\right)^{\frac{\theta p'}{p_1}}\right]^{\frac{p}{p'}}$$

$$= \left(\frac{1}{|Q|}\int_Q w_0^{\frac{-p_0{'}}{p_0}}\right)^{\frac{(1-\theta)p}{p_0}} \left(\frac{1}{|Q|}\int_Q w_1^{\frac{-p_1{'}}{p_1}}\right)^{\frac{\theta p}{p_1}}$$

$$= \left[\left(\frac{1}{|Q|}\int_Q w_0^{\frac{-p_0{}'}{p_0}}\right)^{\frac{p_0}{p_0{}'}}\right]^{\frac{(1-\theta)p}{p_0}} \left[\left(\frac{1}{|Q|}\int_Q w_1^{\frac{-p_1{}'}{p_1}}\right)^{\frac{p_1}{p_1'}}\right]^{\frac{\theta p}{p_1}}.$$

Consequently we get,

$$\left(\frac{1}{|Q|}\int_Q w^{-\frac{p'}{p}}\right)^{\frac{p}{p'}} \leq \left[\left(\frac{1}{|Q|}\int_Q w_0^{\frac{-p_0{}'}{p_0}}\right)^{\frac{p_0}{p_0{}'}}\right]^{\frac{(1-\theta)p}{p_0}} \left[\left(\frac{1}{|Q|}\int_Q w_1^{\frac{-p_1{}'}{p_1}}\right)^{\frac{p_1}{p_1'}}\right]^{\frac{\theta p}{p_1}} -------(4)$$

Now multiplying (3) and (4), we have

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{p'}{p}}\right)^{\frac{p}{p'}} \leq \left(\frac{1}{|Q|}\int_Q w_0\right)^{\frac{(1-\theta)p}{p_0}} \left[\left(\frac{1}{|Q|}\int_Q w_0^{\frac{-p_0{}'}{p_0}}\right)^{\frac{p_0}{p_0{}'}}\right]^{\frac{(1-\theta)p}{p_0}}$$

$$\times \left(\frac{1}{|Q|}\int_Q w_1\right)^{\frac{\theta p}{p_1}} \left[\left(\frac{1}{|Q|}\int_Q w_1^{\frac{-p_1{}'}{p_1}}\right)^{\frac{p_1}{p_1'}}\right]^{\frac{\theta p}{p_1}}$$

$$\leq [w_0]_{A_{p_0}}^{(1-\theta)\frac{p}{p_0}} [w_1]_{A_{p_1}}^{\frac{\theta p}{p_1}}$$

Now taking supremum over all cubes Q in $\boldsymbol{R}^n$, we have $w \in A_p$.

There are other properties of A_p which can be proved using the elementary analysis tools. For more about the weight function and related properties, please refer [2], [3], and [5].

Conclusion

We studied weight functions and we proved some important properties of these weight functions using the elementary analysis tools.

References

1. Banelos R. and Moore C.N, "*Probabilistic Behavior of Harmonic Functions"*, Birkhauser Verlag, USA, (1991).
2. Ghimire S., Two Different Ways to Show a Function is an A_1 Weight Function, "*The Nepali Mathematical Sciences Report"*, 33(1), (2014).
3. Ghimire S., Weighted Inequality, "*Journal of Institute of Engineering"*, 10(1), (2014).
4. Grafakos Loukas, "*Modern Fourier Analysis"*, Second Edition, Springer, (2009).
5. Parajuli V. and Ghimire S., Product of A_p weight functions, "*Journal of Institute of Engineering"*, 11(1), (2015).