A LAW OF THE INTEGRAED LOGARITHM FOR THE TAIL SUMS OF DYADIC MARTINGALES USING STOPPING TIMES

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Abstract

Stopping times have been used in number of places in the derivation of law of iterated logarithm for various context. In this article, we obtain a law of the iterated logarithm for the tail sums of dyadic martingales using stopping times.

Keywords: *Dyadic Martingales, Tail LIL, Stopping times.*

1. Introduction

In probability theory, the law of iterated logarithm (LIL) describes the magnitude of the fluctuation of a random walk. Its study is directly or indirectly related to dyadic interval and dyadic martingales. A dyadic interval of the unit cube [0, 1) is of the form $Q_{nj} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ for $n, j \in \mathbb{Z}$. Generally, we write Q_n to denote a generic interval of length $\frac{1}{2^n}$ [3]. If F_n denotes the σ-algebra generated by the dyadic intervals of the form $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$ on $[0,1)$ then the conditional expectation of f_{n+1} on F_n is given by $E(f_{n+1}|F_n) = \frac{1}{|Q_n|}\int_{Q_n} f_{n+1}(y)dy,\ x \in Q_n$. In this consideration, a dyadic martingale is a sequence of integrable functions $\{f_n\}_{n=0}^{\infty}$ with $f_n: [0,1) \to \mathbb{R}$ such that for every n, f_n is F_n- measurable and $E(f_{n+1}|F_n) = f_n$ for all $n \geq 0$. [2]

For a dyadic martingale, we define the maximal functions as $f_m^* = \sup_{1 \leq k \leq m} |f_k|$ and $f^* = \sup_{1 \leq k < \infty} |f_k|$ and the martingale tail square function is given as ${S'_n}^2 f(x) = \left(S'_n f(x)\right)^2 = \sum_{k=n+1}^{\infty} d_k^2(x)$, where $d_k = f_k(x) - f_{(k-1)}(x)$ is the general term of martingale difference sequence $\{d_k\}_1^{\infty}$. [2]

In addition, for a dyadic martingale, we have $\{x: f^*(x) < \infty\} = \{x: \lim f_n(x) \text{ exists}\}$a.s. [1]

In this context, a theorem on the tail LIL for dyadic martingales gives an important result which is stated in the following theorem.[4]

Theorem 1 (Tail LIL for Dyadic Martingale)

Let $\{f_n\}_{n=0}^{\infty}$ be a dyadic martingale. Assume that there exists a constant $C < \infty$ such that $\left|\frac{S_n' f(x)}{S_n' f(y)}\right| \leq C, \forall x, y \in I_{nj}$ for $n = 1,2,3 \ldots$, $j \in \{0, 1, 2, 3, \ldots, 2^n - 1\}$ where $I_{nj} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$. Then $\limsup_{n \to \infty} \frac{|f_n(x) - f(x)|}{\sqrt{2S_n'^2 f(x) \log\log \frac{1}{S_n'^2 f(x)}}} \leq 2C$ for a. e. x.

From the assumption, we get $Sf(x) < \infty$ for a.e. x. This shows that the sequence $\{f_n(x)\}$ converges [1]. Thus the tail law of the iterated logarithm gives the rate of convergence of dyadic martingales $\{f_n\}$ to its limit function f. Moreover, the rate of convergence depends on the tail sums of martingale square function.

As continuation in the tail LIL for dyadic martingales, we obtained a new result which can be considered as the corollary of the theorem on tail LIL for dyadic martingales stated above. Our main result is as follows.

Theorem 2 Let $\{f_n\}_{n=0}^{\infty}$ be a dyadic martingale. Fix $\theta > 1$. Define stopping times $n_k(x) = \min\left\{n : x \in I_{nj},\ \text{for some } j \in \{1, 2, 3, \ldots, 2^n\} \text{ and } \forall\, y \in I_{nj},\ S_n' f(y) < \frac{1}{\theta^k}\right\}$. Then for the sequence of stopping times $n_k(x)$,

$$\limsup_{k \to \infty} \frac{\left|f(x) - f_{n_k}(x)\right|}{\sqrt{2{S_n'}^2 f(x) \log\log \frac{1}{{S_n'}^2 f(x)}}} < \sqrt{3}$$

for a.e. x.

Proof:

First of all we prove the following estimate for $\lambda > 0$, $\eta > 0$,

$$|\{x \in [0,1) : |f(x) - f_n(x)| > \lambda, S_n' f(x) < \eta\, \lambda\}| \leq \exp\left(\frac{-1}{2\,\eta^2}\right) \quad (1)$$

To prove this we have

$$|\{x : |f(x) - f_n(x)| > \lambda\}| \leq 6 \exp\left(\frac{-\lambda^2}{2\left\|S_n' f\right\|_\infty^2}\right)$$

Here, $S_n' f(x) < \eta\lambda$ gives $||S_n' f||_\infty^2 \leq \eta^2\lambda^2$. So, $\frac{-1}{\left\|S_n' f\right\|_\infty^2} \leq \frac{-1}{\eta^2\lambda^2}$. So we have,

$$|\{x \in [0,1) : |f(x) - f_n(x)| > \lambda, S_n' f(x) < \eta\, \lambda\}| \leq 6 \exp\left(\frac{-\lambda^2}{2\left\|S_n' f\right\|_\infty^2}\right)$$

$$\leq 6 \exp\left(\frac{-\lambda^2}{2\,\eta^2\lambda^2}\right)$$

$$= \exp\left(\frac{-1}{2\,\eta^2}\right)$$

This is the required result (1).

Now, choose $\lambda = \frac{(1+\epsilon)\sqrt{2\log\log\theta^{2l}}}{\theta^2}$ and $\eta = \frac{\theta}{(1+\epsilon)\sqrt{2\log\log\theta^{2l}}}$ where $\theta > 1$ and $\epsilon > 0$. Then using (1) we have,

$$\left|\left\{x \in [0,1): |f(x) - f_n(x)| > \frac{(1+\epsilon)\sqrt{2\log\log\theta^{2l}}}{\theta^2}, S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right|$$

$$\leq 6\exp\left(\frac{-(1+\epsilon)^2(2\log\log\theta^{2l}}{2\theta^2}\right)$$

$$= 6\exp\left(\log\,(2l\log\theta)^{\frac{-(1+\epsilon)^2}{\theta^2}}\right)$$

$$= 6\,(2l\log\theta)^{\frac{-(1+\epsilon)^2}{\theta^2}}$$

$$= \frac{6}{(2l\log\theta)^{\frac{(1+\epsilon)^2}{\theta^2}}}$$

$$= \frac{6}{(2\log\theta)^{\frac{(1+\epsilon)^2}{\theta^2}}}\cdot\left(\frac{1}{l}\right)^{\frac{(1+\epsilon)^2}{\theta^2}}$$

Let us choose $\epsilon = \sqrt{3}\,\theta - 1$. Then we have $\frac{(1+\epsilon)^2}{\theta^2} = 3$. Thus,

$$\left|\left\{x \in [0,1): |f(x) - f_n(x)| > \frac{(1+\epsilon)\sqrt{2\log\log\theta^{2l}}}{\theta^2}, S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right| \leq 6\left(\frac{1}{2\log\theta}\right)^3 . \frac{1}{l^3}$$

$$= \frac{C}{l^3} \text{ (suppose).} \qquad (2)$$

Now, let $(x) = \sqrt{x\log\log\frac{1}{x}}$. Then $g(x)$ is an increasing function. So for $\frac{1}{\theta^{2l}} \leq {S_n'}^2 f(x)$, we have,

$$\sqrt{2{S_n'}^2 f(x)\log\log\frac{1}{{S_n'}^2 f(x)}} \geq \sqrt{2\frac{1}{\theta^{2l}}\log\log\theta^{2l}} \qquad (3)$$

Now, using (3), we have,

$$\left|\left\{x \in [0,1): |f(x) - f_n(x)| > (1+\epsilon)\sqrt{2{S_n'}^2 f(x)\log\log\frac{1}{{S_n'}^2 f(x)}}\right\}\right|$$

$$= \left|\cup_{l=k+1}^{\infty}\left\{x \in [0,1): |f(x) - f_n(x)| > (1+\epsilon)\sqrt{2{S_n'}^2 f(x)\log\log\frac{1}{{S_n'}^2 f(x)}}, \frac{1}{\theta^l} \leq S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right|$$

$$\leq \left|\cup_{l=k+1}^{\infty}\left\{x \in [0,1): |f(x) - f_n(x)| > (1+\epsilon)\sqrt{2\frac{1}{\theta^{2l}}\log\log\theta^{2l}}, S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right|$$

$$= \left|\cup_{l=k+1}^{\infty}\left\{x \in [0,1): |f(x) - f_n(x)| > \frac{1+\epsilon}{\theta^l}\sqrt{2\log\log\theta^{2l}}, S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right|$$

$$\leq \sum_{l=k+1}^{\infty}\left|\left\{x \in [0,1): |f(x) - f_n(x)| > \frac{1+\epsilon}{\theta^l}\sqrt{2\log\log\theta^{2l}}, S_n' f(x) < \frac{1}{\theta^{l-1}}\right\}\right|$$

$\leq \sum_{l=k+1}^{\infty} \frac{C}{l^3}$ (4)

We know that,

$$\sum_{l=k+1}^{\infty} \frac{1}{l^3} \leq \int_k^{\infty} \frac{1}{x^3}\,dx = \left[\frac{-1}{2x^2}\right]_k^{\infty} = \frac{1}{k^2}$$

So, (4) can be written as,

$$\left|\left\{x \in [0,1): |f(x) - f_n(x)| > (1+\epsilon)\sqrt{2{S'_n}^2 f(x) \log\log \frac{1}{{S'_n}^2 f(x)}}\right\}\right| \leq \frac{C}{k^2}$$

This can be done for every $n_k(x)$. So summing over all k we have,

$$\sum_{k=1}^{\infty}\left|\left\{x \in [0,1): |f(x) - f_n(x)| > (1+\epsilon)\sqrt{2{S'_n}^2 f(x) \log\log \frac{1}{{S'_n}^2 f(x)}}\right\}\right| \leq \sum_{k=1}^{\infty} \frac{C}{k^2}$$

$$= C \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

So, by Borel Cantelli lemma, for a.e. x, there exists M which depends on x such that for every $k \geq M$,

$$\left|f(x) - f_{n_k}(x)\right| \leq (1+\epsilon)\sqrt{2{S'_n}^2 f(x) \log\log \frac{1}{{S'_n}^2 f(x)}}$$

But we have choosen $\epsilon = \sqrt{3}\,\theta - 1$. So,

$$\left|f(x) - f_{n_k}(x)\right| \leq \sqrt{3}\,\theta\sqrt{2{S'_n}^2 f(x) \log\log \frac{1}{{S'_n}^2 f(x)}}$$

that is,

$$\frac{\left|f(x) - f_{n_k}(x)\right|}{\sqrt{2{S'_n}^2 f(x) \log\log \frac{1}{{S'_n}^2 f(x)}}} \leq \sqrt{3}\,\theta$$

It is noted that as $n \to \infty, k \to \infty$. Now, letting $\theta \downarrow 1$, we get for a. e. x,

$$\limsup_{k \to \infty} \frac{\left|f(x) - f_{n_k}(x)\right|}{\sqrt{2{S'_n}^2 f(x) \log\log \frac{1}{{S'_n}^2 f(x)}}} \leq \sqrt{3}$$

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