DISCRETE MAXIMUM PRINCIPLE IN ONE-DIMENSIONAL HEAT EQUATION

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Abstract

The maximum principle plays key role in the theory and application of a wide class of real linear partial differential equations. In this paper, we introduce 'Maximum principle and its discrete version' for the study of second-order parabolic equations, especially for the one-dimensional heat equation. We also give a short introduction of formation of grid as well as finite difference schemes and a short prove of the 'Discrete Maximum principle' by using different schemes of heat equation.

Keywords: *Grid, Finite Difference Scheme, Finite Difference Methods, Heat Equation, Uniqueness.*

1. Introduction

In the early 1800s, J. Fourier began a mathematical study of heat equation. The heat equation is

$$u_t - \Delta u = 0$$

And non-homogeneous heat equation is

$$u_t - \Delta u = f$$

at point x in time t and $u: \overline{U} \times [0,\infty) \to \mathbb{R}$, $u = u(x,t)$ and the given function $f \to U \times [0,\infty) \to \mathbb{R}$ and $\Delta = \sum_{i=1}^{n} u_{x_i u_i}$ is Laplacian [3].

If $x \in U$ and $U \subset \mathbb{R}$, then one-dimensional heat equation is

$u_t(x,t) = b\, u_{xx}(x,t)$ for $b > 0$ … … … (1)

By using the Fourier transform and Fourier inversion formula the solution of one-dimensional heat equation

$$u_t(x,t) = b\, u_{xx}(x,t) \text{ for } b > 0, \qquad x \in \mathbb{R}, \qquad t \geq 0$$

$$u(x,0) = u_0(x)$$

is

$$u(x,t) = \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4bt}} u_0(y)dy$$

Where $u(x,t)$ gives the temperature at time t and point x [7].

2. Finite Difference Grids and Schemes

The close solution domain $D(x,t)$ is xt-plane for a two dimensional equilibrium problem as shown in figure (1). The solution domain must be covered by two dimensional grids of lines, called the finite difference grid. The finite difference solution to the PDE is obtained at the intersections of these grid lines.

Let these grid lines be equally spaced lines perpendicular to the x and t directions having uniform spacing h and k. Discretize horizontally into m equal intervals h and vertically into n equal intervals k where h and k are positive numbers, n and m are arbitrary integers. Set of grid points are denoted by$(x_m, t_n) = (mh,\ nk)$. On the grid point, (x_m, t_n) a continuous function $u(x, t)$ which is varying on (x, t) is denoted by u_m^n. A grid function v defined by on the grid point (x_m, t_n) is denoted by v_m^n. The value of functions u and v are the same on the grid points [5,7].

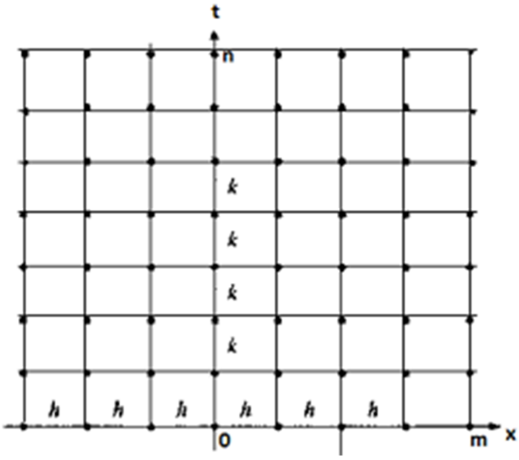

Fig1 The Finite Difference Grid

2.1 Finite Difference Method

The finite difference method is one of several techniques for obtaining numerical solution to PDEs. If we replace partial derivatives of PDEs by finite difference by using Taylor's series, we obtain the finite difference schemes of PDEs. Aim of finite difference schemes is to approximate the values of the continuous function $u(x, t)$ on a set of grid points in xt-plane. The finite difference method was first developed by A. Thom in 1920s under the title 'The method of square' to solve non-linear hydrodynamic equations. The finite difference method is one of the numerical approximation methods that solve finite difference schemes by using iteration methods or by using computational algorithm to obtain an approximate solution of the PDE.

2.2 Some important finite difference schemes of heat equation:

The forward and backward difference operator are defined by

$\delta_+ v_m = \frac{v_{m+1} - v_m}{h}$ and $\delta_- v_m = \frac{v_m - v_{m-1}}{h}$ respectively.

The difference operator is defined by

$\delta_0 = \frac{1}{2}\ (\delta_+ v_m + \delta_- v_m)$ (For first-order derivative)

$\delta^2 v_m = \left(\frac{\delta_+ - \delta_-}{h}\right)(v_m) = \frac{v_{m+1} - 2v_m + v_{m-1}}{h^2}$ (For second-order derivative)

Consider each grid point separated by a distance h along x-direction. By Taylor's series expansion of $u(x + h, t)$and $u(u - h, t)$ and adding both of them, we obtain

$$\frac{\partial^2}{\partial x^2}\ u(x, t) = \frac{u\ (x + h, t) + u(x - h, t) - 2u(x, t)}{h^2} + o(h^2)$$

This is called central difference approximation.

On the grid point, $u_m^n = v_m^n$ so that for the construction of finite difference schemes we use grid function v by replacing u because schemes for solving PDEs, we are restricted to grid point.

2.3 The forward-time central-space scheme:

The forward-time central-space scheme for the heat equation (1) is

$$\frac{u(x_m,\ t_n+k)-u\,(x_m,u_t)}{k}=b\,\frac{u(x_m+h,\ t_n)-2u\,(x_m,u_t)+\ u(x_m-h,\ t_n)}{h^2}$$

$$\Rightarrow \frac{u_m^{n+1}-u_m^n}{k}=b\,\frac{u_{m+1}^n-2u_m^n+\ u_{m-1}^n}{h^2}$$

Since on grid points $u_m^n = v_m^n$, we obtain

$$\frac{v_m^{n+1}-v_m^n}{k}=b\,\frac{v_{m+1}^n-2v_m^n+\ v_{m-1}^n}{h^2}$$

Similarly,

2.4 The backward-time and central-space scheme:

The backward-time and central-space scheme for the equation (1) is

$$\frac{v_m^{n+1}-v_m^n}{k}=b\,\frac{v_{m+1}^{n+1}-2v_m^{n+1}+\ v_{m-1}^{n+1}}{h^2}$$

2.5 The Crank-Nicholson scheme:

The Crank-Nicholson scheme for the heat equation (1) is

$$\frac{v_m^{n+1}-v_m^n}{k}=\frac{b}{2}\frac{v_{m+1}^{n+1}-2v_m^{n+1}+v_{m-1}^{n+1}}{h^2}+\frac{b}{2}\frac{v_{m+1}^n-2v_m^n+v_{m-1}^n}{h^2}$$

Where $u_t$is approximated with backward time-difference and u_{xx} is approximated with the average of the central difference schemes evaluated at the current and the previous time steps.

Discrete Maximum Principle

The usefulness of maximum principle is restricted to second-order equation because second-order derivatives of a function give information of the function at extrema. The maximum principle and its discrete version are used in pure as well as applied mathematics to find the maximum and minimum values of the continuous as well as discrete functions. It is an important property of parabolic equations used to deduce a variety of results such as uniqueness, boundedness comparison principles. The maximum and minimum values of the function lie on the boundary of any domain, for example , in a steady temperature distribution, both hottest and coldest temperature occurs at the boundary of the region, as shown in figure (2).

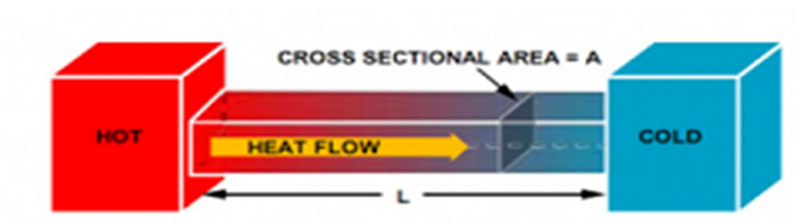

Fig 2 Steady Temperature Distribution

The discrete function v_m^n plays the same role on a region as the continuous function $u(t,x)$ plays role to find the maximum and minimum values i.e. the discrete maximum principle is used to find the maximum and minimum values of the discrete functions. Hence, discrete maximum principle is the discrete counterpart of the maximum principle [1,2].

Theorem (Maximum Principle)

Let $Q_T: 0 \le x \le l, 0 \le t \le T$ be a closed region and Γ_T: Boundary of Q_T i.e. $x = 0, t = 0$ or $x = l$, but $t \neq T$. Let $u(x,t)$ be a solution of

$$u_t = \alpha^2 u_{xx}, \quad (x,t)\epsilon Q_T$$

which is continuous in the closed region Q_T. The maximum and minimum values of $u(x,t)$ are on the boundary Γ_T (Initial line $t = 0$ at the points on the boundary $x = 0$ or $x = l$) [8]

Theorem (Discrete Maximum Principle)

i. The forward-time central-space scheme (Explicit) for the heat equation

$$u_t(x,t) = b\, u_{xx}(x,t) \text{ for } b > 0$$

is

$$\frac{v_m^{n+1} - v_m^n}{k} = b\,\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} \quad \dots\dots\dots (2)$$

Show that

$$\max_m |v_m^{n+1}| \le \max_m |v_m^n| \text{ for the CFL condition } \mu = \frac{bk}{h^2} \le \frac{1}{2} \quad [7]$$

ii. The backward-time central-space scheme (Implicit) for the heat equation

$$u_t(x,t) = b\, u_{xx}(x,t) \text{ for } b > 0$$

is

$$\frac{v_m^{n+1} - v_m^n}{k} = b\,\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} \quad \dots\dots\dots (3)$$

Show that

$$\max_m |v_m^{n+1}| \le \max_m |v_m^n| \text{ for all} \mu = \frac{bk}{h^2} > 0 \quad [7]$$

iii. The Crank-Nicolson Implicit scheme for the heat equation

$$u_t(x,t) = b\, u_{xx}(x,t) \text{ for } b > 0$$

is

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{b}{2}\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2} + \frac{b}{2}\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} \quad \dots\dots\dots (4)$$

Show that

$$\max_m |v_m^{n+1}| \le \max_m |v_m^n| \text{ for all } 0 < \mu = \frac{bk}{h^2} \le 1 \quad [7]$$

Proof:

i. The forward-time central-space explicit scheme (2) can be written as

$$v_m^{n+1} = v_m^n + \mu\,(v_{m+1}^n - 2v_m^n + v_{m-1}^n) \text{where} \mu = \frac{bk}{h^2}$$

$$\Rightarrow v_m^{n+1} = \mu\, v_{m+1}^n + \mu\, v_{m-1}^n + (1 - 2\mu) v_m^n \dots\dots\dots (5)$$

For m = 1,2,3,…, M - 1 and n= 0, 1, 2, 3, ……, to final computing time.

We have, CFL condition $\mu \leq \frac{1}{2}$i.e$1-2\mu \geq 0$.

Now from, we have

$$|v_m^{n+1}| \leq \mu|v_{m+1}^n| + \mu|v_{m-1}^n| + (1-2\mu)|v_m^n|$$
$$\leq \max_i |v_i^n| + \mu \max_i |v_i^n| + (1-2\mu)\max_m |v_i^n|$$

Where i is specific number.

$$|v_m^{n+1}| \leq \max_i |v_i^n|$$

It holds for m when v_m^{n+1} reaches maximum.

$\therefore \max_m |v_m^{n+1}| \leq \max_m |v_m^n|$ for all m=1, 2, 3, … … … , M – 1

which is discrete maximum principle.

ii. The back-time central-space explicit scheme (3) can be written as

$v_m^{n+1} = v_m^n + \mu(v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n-1})$ where $\mu = \frac{bk}{h^2}$

$\Rightarrow (1-2\mu)v_m^{n+1} = v_m^n + \mu v_{m+1}^{n+1} + \mu v_{m-1}^{n-1}$ … … (6)

form = 1,2,3,…, M - 1 and n = 0, 1, 2, 3, ……, to final computing time.

Since$\mu < 0$, we hav

$$(1+2\mu)|v_m^{n+1}| \leq |v_m^n| + \mu|v_{m+1}^{n+1}| + \mu|v_{m-1}^{n+1}|$$
$$\leq \max_i |v_i^n| + \mu \max_i |v_i^{n+1}| + \mu \max_m |v_i^{n+1}|$$

Where i is specific number. Then

$$|v_m^{n+1}| \leq \max_i |v_i^n|$$

It holds for m when v_m^{n+1} reaches maximum.

$$\therefore \max_m |v_m^{n+1}| \leq \max_m |v_m^n| \text{ for all } m=1, 2, 3, \ldots \ldots \ldots, M-1$$

which is discrete maximum principle.

The proof of (iii) is similar to (ii).

3. Conclusion

In mathematical modeling of the real world problems, non-linear or non-homogeneous PDEs or mixed type boundary conditions as well as time dependent boundary conditions, we use numerical method by discretizing the PDEs into finite difference schemes to get approximate solution with minimum error. The discrete maximum principle is useful in the resulting matrix equations, which approximate parabolic boundary value problem by employing the finite difference method. The discrete maximum principle is applied not only to linear boundary value problems, but also to non-linear boundary value problem in scientific and engineering disciplines.

Hence, we conclude that in the various PDEs and real world problems, numerical solution methods are better in comparison to the explicit solution methods and the discrete maximum principle is very useful in non-linear or non-homogeneous PDEs to find the extreme values on the boundary of the domain.

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