# SOME NEW TYPES OF ORTHOGONALITIES IN NORMED SPACES AND APPLICATION IN BEST APPROXIMATION

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**Abstract**

In this paper, two new types of orthogonality from the generalized Carlssion orthogonality have been studied and some properties of orthogonality in Banach spaces are verified and as best implies Birkhoff orthogonality and Birkhoff orthogonality implies best approximation, in this paper, Pythagorean orthogonality also implies best approximation has been proved.

***Keywords:*** *Birkhof orthogonality, James orthogonality, Carlsson orthogonality, Pythagorean orthogonality, Best approximation.*

## 1. Introduction

Orthogonality is one of the most important concepts in the theory of inner product spaces. Let X be an inner product space. Then a vector $x \in X$ is said to be orthogonal to the vector $y \in X$ if $(x, y) = 0$. However, when switching from the theory of inner product spaces to normed linear spaces, we cannot maintain the notion of orthogonality. Therefore, since 1934 generalized orthogonality types for normed linear spaces have been introduced and studied. When the norm of X is induced by an inner product, the orthogonality in X is equivalent to each of the following propositions. This motivates that in context of normed linear spaces, any one of the following proposition can be taken as a definition of orthogonality in X.

**Roberts (1934):** In a normed linear space X,

$$x \perp y \Leftrightarrow \forall \alpha \ \|x + \alpha y\| = \|x + \alpha y\|$$

**Birkhoff (1935):** In a normed linear space X,

$$x \perp y \Leftrightarrow \forall \alpha \ \|x\| \leq \|x + \alpha y\|$$

**Isosceles (1945):** In a normed linear space X,

$$x \perp y \Leftrightarrow \forall \alpha \ \|x - y\| = \|x + y\|$$

**Pythagorean (1945):** In a normed linear space X,

$$x \perp y \Leftrightarrow \forall \alpha \ \|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

**Carlssion (1961):** In a normed linear space X,

$x \perp y \Leftrightarrow \sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0$, where $m \geq 2$ and $a_k$, $b_k$, $c_k$ are real numbers such that

$$\sum_{k=1}^{m} a_k b_k c_k \neq 0, \sum_{k=1}^{m} a_k {b_k}^2 = \sum_{k=1}^{m} a_k {c_k}^2 = 0.$$

### 1.1 Main properties of orthogonality in inner product spaces

The following are the main property of orthogonality in inner product spaces (see [9]). Let (X, ⟨.⟩) be an inner product space and x, y, z ∈ X. Then,

(i) if $x \perp x$, then x = 0 (non-degeneracy);
(ii) if $x \perp y$, then $\lambda x \perp \lambda y$ for all $\lambda \in R$ ( simplification);
(iii) if $(x_n)$,$(y_n) \subset X$ such that $(x_n) \perp (y_n)$ for every $n \in N$, $x_n \to x$, $y_n \to y$, then $x \perp y$ (continuity);
(iv) if $x \perp y$, then $\lambda x \perp \mu y$ for all $\lambda, \mu \in R$ (homogeneity);
(v) if $x \perp y$ , then $y \perp x$ (symmetry);
(vi) if $x \perp y$ and $x \perp z$, then $x \perp y + z$ (additivity);
(vii) if $x \neq 0$, then there exist $\lambda \in R$ such that $x \perp \lambda x + y$ (existence);
(viii) the above $\lambda$ is unique. (uniqueness)

**Definition:** Let X be a normed linear space, and G be a nonempty subset of X. An element $g_0 \in G$ is called a best approximation to x from G if

$$PG(x) = \{g_0 \in G : ||x - g_0|| \leq ||x - g|| \text{ for all } g \in G\} .$$

**Definition:** If PG(x) contains at least one element, then the subset G is called a proximinal set. In other words, if $PG(x) \neq \varphi$ then G is called proximinal set.

## 2. Main Result

The Roberts, Pythagorean and Isosceles orthogonalities have been generalized by Carlssion in 1961 [4]. Furthermore, these orthogonalities are obtained by giving particular values of constants in a generalized Carlssion's orthogonality. In this section we will show that how two other new orthogonality relations can be obtained from Carlssion orthogonality and application of orthogonality in best approximation.

### 2.1 New orthogonality- I

**Definition:** A vector x in X is said to be orthogonal to a vector y in X if and only if

$$2||x + y||^2 + i\,||x + iy||^2 = 2||x - y||^2 + i||x - iy||^2.$$

This is a particular case of Carlssion's orthogonality which is obtained by assigning particular values to constants $a_k$, $b_k$ and $c_k$as follows.

$$a_1 = b_1 = c_1 = 1,\ a_2 = \frac{i}{2},\ b_2 = 1,\ c_2 = i,\ a_3 = -1,\ b_3 = 1,\ c_3 = -1,\ a_4 = -\frac{i}{2},\ b_4 = 1,\ c_4 = -i.$$

The above values of constants satisfy Carlssion's condition

$$\sum_{k=1}^{4} a_k b_k c_k = 1,\ \sum_{k=1}^{4} a_k {b_k}^2 = \sum_{k=1}^{4} a_k {c_k}^2 = 0.$$

### 2.2 New orthogonality- II

**Definition:** A vector x in X is orthogonal to a vector y in X if and only if

$$||x + \frac{1}{2}y||^2 + ||x - \frac{1}{2}y||^2 = \frac{1}{2}||\sqrt{2}x + y||^2 + ||x||^2.$$

This is also a particular case of Carlssion's orthogonality. For this, we take values of constants as;

$$a_1 = a_2 = 1,\ a_3 = a_4 = -\frac{1}{2},\quad b_2 = 1 = b_1,\ b_3 = \sqrt{2} = b_4,\ c_1 = \frac{1}{2},\ c_2 = -\frac{1}{2},\ c_3 = 1, \text{ and } c_4 = 0.$$

Satisfying the condition $\sum_{k=1}^{4} a_k b_k c_k = -\frac{1}{\sqrt{2}}, \sum_{k=1}^{4} a_k {b_k}^2 = \sum_{k=1}^{4} a_k {c_k}^2 = 0$.

**Lemma 2.1:** Let X be a real inner product space. If $\|x+\frac{1}{2}y\|^2 + \|x-\frac{1}{2}y\|^2 = \frac{1}{2}\|\sqrt{2}x+y\|^2 + \|x\|^2$ for all x, y ∈ X, then $x \perp y$

**Proof:** Let x, y ∈ X and suppose that.

$$\|x+\frac{1}{2}y\|^2 + \|x-\frac{1}{2}y\|^2 = \frac{1}{2}\|\sqrt{2}x+y\|^2 + \|x\|^2.$$

$$\Rightarrow \quad 2\|x\|^2 + \frac{1}{2}\|y\|^2 = \sqrt{2}\langle x, y\rangle + 2\|x\|^2 + \frac{1}{2}\|y\|^2$$

$$\Rightarrow \quad \sqrt{2}\langle x, y\rangle = 0$$

$$\Rightarrow x \perp y.$$

### 2.2.1 Properties of orthogonality relation of type I and II.

1. Orthogonality relation $2\|x+y\|^2 + i\|x+iy\|^2 = 2\|x-y\|^2 + i\|x-iy\|^2$ satisfies non-degeneracy, simplification and continuity.

2. Orthogonality relation $\|x+\frac{1}{2}y\|^2 + \|x-\frac{1}{2}y\|^2 = \frac{1}{2}\|\sqrt{2}x+y\|^2 + \|x\|^2$ also satisfies non-degeneracy, simplification and continuity and homogeneous if the space is an inner product space.

### 2.3 Application of Orthogonality in Best Approximation

**Theorem 2.2:** [11] Let X be a normed space and G be a subspace of X. Then, $g_0 \in PG(x)$ if and only if $(x - g_0) \perp BG$.

The above theorem shows that the Birkhoff orthogonality implies best approximation and best approximation implies Birkhoff orthogonality. Besides that Robert orthogonality, Pythagorean orthogonality and Isosceles orthogonality also implies best approximation.

**Lemma 2.3:** Let X normed space. If $\forall x \in X \; \exists \; g_0 \in G$: $x - g_0 \perp P\,(G)$ then $g_0 \in PG(x)$.

**Proof:** Suppose $x - g_0 \in \perp P\, y$. Then

$$\|x - g_0 - y\|^2 = \|x - g_0\|^2 + \|y\|^2$$

$$\Rightarrow \|x - g_0\|^2 \leq \|x - g_0 - y\|^2$$

$$\Rightarrow \|x - g_0\| \leq \|x - g\|, \text{ where } g = g_0 + y$$

$$\Rightarrow g_0 \in PG(x).$$

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