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Parametric Instability in Mathieu Equation in Earthquake Dynamics

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Abstract: *We propose an study of parametric resonance between P-waves and S-waves, which can be used to describe various nonlinear phenomena qualitatively and to obtain bifurcation diagrams quantitatively. We have shown that it is a good simulation of parametric phenomena, and our results are in good agreement with theoretical predictions. In particular, it may be used to study the influence of pump P waves on the instability's threshold and amplitude of S waves in earthquake phenomena that could be simulated with an electronic model.*

Keywords: Mathieu equation, Earthquake dynamics, p-waves and s-waves

1. INTRODUCTION

The difference in speed of travel of P-waves and S-wave is vital to transmit energy of seismic wave; the P wave is a *longitudinal* wave or a *compression* wave. Force is applied in the direction that the wave is travelling; ground or earth is pretty incompressible, so the energy is transferred pretty quickly. In S wave, the medium is displaced in a *transverse* (up and down - compared to the line of travel) way, and the medium must *shear* or "move away" from the material right next to it to cause the shear and transmit the wave [1, 2].

On the other hand, parametric amplifiers and oscillators have been widely studied in electronics and optics. For instance, parametric amplification has been used to achieve low-noise amplification in electronic systems; this parametric instability is called the Faraday instability. Recently, Pritchett and Kim proposed a simple system to observe Faraday instabilities [3]. We propose here an alternative way to study parametric instabilities by doing an analog experiment that models the Mathieu equation [4]. A study can be made on the torsional-lateral motions of non-linear symmetrical structures subjected to lateral ground motion. The torsional and lateral response of a single mass symmetrical system subjected to sinusoidal ground motion can be investigated where non-linear coupling exists between the lateral and rotational motions like S waves. For sinusoidal lateral response, the torsional motion equation can be cast in the form of a Mathieu equation. The likelihood of induced torsional response can be studied in terms of unstable regions in the parametric amplitude-frequency parameter space. The implication of this type of non-linear torsional-lateral coupling to the responses of real symmetrical structures subjected to actual earthquake ground motion can be simulated with an electronic model.

Our proposed simulation is easy to understand conceptually and has several advantages, including fast data acquisition. It also allows students to explore the behavior of a driven oscillator and to understand the concepts of supercritical and subcritical bifurcations in earthquake phenomena. The experiments involve two control parameters: the forcing amplitude P and the forcing pulsation ω_e ; here, for wave P we make $\omega_e = 2\text{rad}/\text{s}$ Varying these two parameters allows us to study the threshold of the instability for different driving frequencies and to explore the Mathieu extension of the bifurcation. We also study the nonlinear dependence of the oscillation amplitude on P .

2. PARAMETRIC RESONANCE

A system is subjected to a parametric forcing if one of its parameters is temporally modulated producing parametric instability which occurs when a tank containing a liquid is vertically vibrated: one then observes standing waves on the free surface [5]. This parametric instability is called the Faraday instability. The physical situation is more complex because of the great number of degrees of freedom in the system leading to the generation of complex patterns on the surface. The study of parametric surface waves has led to a large number of theoretical and experimental studies [6].

In the simple case of an incompressible, irrotational, and inviscid fluid, Benjamin and Ursell [7], showed that, in the linear approximation, each mode ξ_k (of wave vector **k)** of the surface deformation is governed by a Mathieu equation:

$$\frac{d^2\xi_k}{dt^2}+\lambda\frac{d\xi_k}{dt}+\omega_0^2(1+F\cos(\omega_e t)\xi_k=0, \quad (1)$$

which can be extended to rotational motion:

$$\frac{d^2\xi_\theta}{dt^2}+\lambda\frac{d\xi_\theta}{dt}+\omega_0^2(1+P\cos(\omega_e t)\xi_\theta=0, \quad (2)$$

where ξ_θ is transversal. This system leads to a canonical example of a parametric instability [5], ω_e is the external forcing pulsation, and P is directly related to the amplitude of the vibration acceleration relative to the acceleration of gravity produced by P- waves. The presence of a small viscous dissipation can be taken into account by including a phenomenological damping term λ.

In the undamped case $\lambda=0$, when $P\to 0$, the parametric resonance occurs when $\omega_e/\omega_0=2/n$, where n is an integer. The most unstable oscillation corresponds to $n=1$, that is, $\omega_e/\omega_0=2$, [8-11]; in the following, we are interested only in this last case. With $\lambda=0$ and $\omega_0^2P=\varepsilon$ the Mathieu equation (2) is given by:

$$\frac{d^2\xi_\theta}{dt^2}+(\omega_0^2+\varepsilon\cos(2t))\xi_\theta=0 \ , \ \omega_0^2>0 \quad (3)$$

which is reversible. A typical question is: for which values of ω_0 and ε in (ω_0^2,ε)-parameter space is the trivial solution $\xi_\theta=0, d\xi_\theta/dt=0$ stable? We find that periodic solutions exist for *n* = 1 if $\omega_0^2=1\pm\varepsilon/2+O(\varepsilon^2)$; in the case *n* = 2, periodic solutions exist if $\omega_0^2=4-\varepsilon^2/48+O(\varepsilon^4)$, and $\omega_0^2=4+\varepsilon^2 5/48+O(\varepsilon^4)$. The corresponding instability domains are called Floquet tongues, instability tongues or resonance tongues, see Fig. 1.

On considering higher values of *n*, we have to calculate to a higher order of ε. At *n* = 1 the boundary curves are intersecting at positive angles at $\varepsilon=0$, at *n* = 2 ($\omega_0^2=4$) they are tangent; the order of tangency increases as *n* − 1 (contact of order *n*), making instability domains more and more narrow with increasing resonance number *n*.

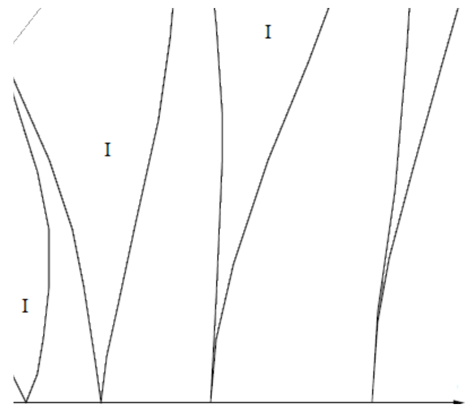

Figure 1. Floquet tongues of the Mathieu eq. (3); the instability domains are marked with I.

3. THE MATHIEU EQUATION WITH VISCOUS DAMPING

In real seismic applications there is always the presence of damping. We shall consider the effect of its simplest form, small viscous damping, eq. (3) is extended by adding a linear damping term λ :

$$\frac{d^2\xi_\theta}{dt^2}+\lambda\frac{d\xi_\theta}{dt}+\omega_0^2(1+P\cos(2t)\xi_\theta=0, \ \lambda>0. \quad (4)$$

We assume that the damping coefficient is small, $\lambda=\varepsilon\kappa_0$, and we put $\omega_0^2=n^2-\varepsilon\beta$ to apply the Poincaré-Lindstedt method [9]. We find periodic solutions in the case *n* = 1 if:

$$\omega_0^2=1\pm\sqrt{\varepsilon^2/4-\lambda^2}\ ; \quad (5)$$

the relation (5) corresponds with the curve of periodic solutions, which in (ω_0^2,ε)-parameter space separates stable and unstable solutions. We observe the following phenomena.

If $0<\lambda<\varepsilon/2$, we have an instability domain which by damping has been lifted from the ω_0^2-axis; also the width has shrunk. If $\cdot\lambda>\varepsilon/2$ the instability domain has vanished. For an illustration see Fig. 2.

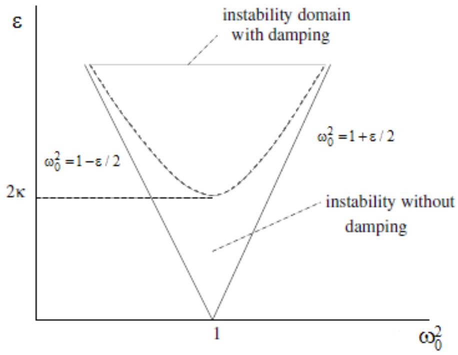

Figure 2. Reduced schematic instability with the damping presence.

Repeating the calculations for $n > 2n$ ¸ 2 we find no instability domains at all; damping of $O(\varepsilon)$ stabilizes the system for ε small. To find an instability domain we have to decrease the damping, for instance if $n = 2$ we have to take

$\lambda = \varepsilon^2 \kappa_0$.

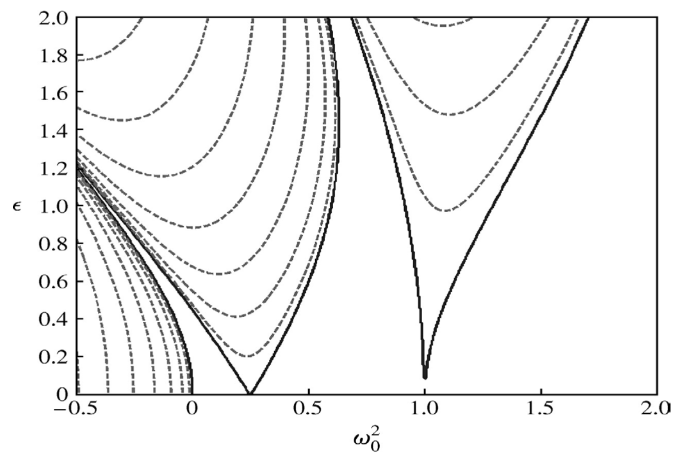

Figure 3. $(\omega_0^2, \varepsilon)$ space for different values of damping.

4. CONCLUSION

We have proposed a study of parametric resonance between P-waves and S-waves, which can be used to describe various nonlinear phenomena qualitatively and to obtain bifurcation diagrams quantitatively. We have shown that it is a good simulation of parametric phenomena, and our results are in good agreement with theoretical predictions. In particular, it may be used to study the influence of pump P waves on the instability's threshold and amplitude of S waves in earthquake phenomena. The implication of this type of non-linear torsional-lateral coupling to the responses of real symmetrical structures subjected to actual earthquake ground motion could be simulated with an electronic model.

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