A Note on the Riccati Equation

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Abstract: *We show the relationship between the solutions of two Riccati equations associated to Schrödinger equations interconnected via a Darboux transformation.*

Keywords: riccati equation, schrödinger equation, darboux transformation, energy spectrum in quantum mechanics

1. INTRODUCTION

The Schrödinger equation permits to study a quantum system via the wave function, besides this equation leads to the energy levels for a given potential. Then it is natural to ask if this quantum information can be employed for another system with a different potential but with the same energy spectrum. The answer is yes if both systems are interconnected through a Darboux transform, and in this process the Riccati equation has an important role. Therefore, if two Schrödinger equations have a Darboux type relationship then it is useful to investigate the behaviour of the corresponding non-linear Riccati equations. Our analysis shows that a Darboux transformation adds an exact derivative to the Riccati solution, and we consider that this fact explains why the energy values are invariant under Darboux mapping.

2. SCHRÖDINGER AND RICCATI EXPRESSIONS

The Schrödinger equation:

$$\Psi'' + (\lambda - u)\Psi = 0 \quad , \tag{1}$$

where Ψ, u and λ represent the wave function, the potential and the eigenvalue of energy, respectively, can be transformed to a Riccati equation (name introduced by D'Alembert in 1763), Riccati 1724, Kryachko 2005:

$$R + R^2 + \lambda - u = 0 \quad , \tag{2}$$

employing the following change in the dependent variable, Young 1931, Lanczos 1997, Kryachko 2005:

$$R = \frac{\Psi'}{\Psi} \quad . \tag{3}$$

If now a Darboux transformation, Darboux 1882, Matveev and Salle 1991, López-Bonilla, Morales and Ovando 2002, Caltenco, López-Bonilla and Acevedo 2004, is applied to (1) then a Schrödinger equation with the same energy levels, but with different wave functions Φ and another potential U, is obtained:

$$\Phi'' + (\lambda - U)\Phi = 0 \quad , \tag{4}$$

such that:

$$\Phi = \Psi' - \Psi\sigma_1 \quad , \quad U = u - 2\sigma_1' \quad , \quad \sigma_1 = \frac{\Psi_1'}{\Psi_1} \quad , \tag{5}$$

being Ψ_1 the solution of (1) for λ_1:

$$\Psi_1'' + (\lambda_1 - u)\Psi_1 = 0 \quad . \tag{6}$$

Then (4) has associated the Riccati equation:

$$\tilde{R}' + \tilde{R}^2 + (\lambda - U) = 0 \quad , \quad \tilde{R} = \frac{\Phi'}{\Phi} \quad , \tag{7}$$

and it is natural to ask on the relationship between R and $\tilde{R}$.

From (5):

$$\Phi = \Psi' - \Psi\sigma_1 \overset{(3)}{=} \Psi R - \Psi\sigma_1 = \Psi(R - \sigma_1) \quad ,$$

which can be used in (7):

$$\tilde{R} = \frac{1}{\Psi(R - \sigma_1)}\left[(R - \sigma_1)\Psi' + \Psi(R - \sigma_1)'\right] = \frac{\Psi'}{\Psi} + \frac{(R - \sigma_1)'}{R - \sigma_1} \quad ,$$

and thus to obtain an interesting expression between the Riccati functions:

$$\tilde{R} = R + \frac{d}{dx}\text{Ln}(R - \sigma_1) \quad , \tag{8}$$

when their associated Schrödinger equations are connected via a Darboux transformation. The relations:

$$R'' + 2RR' = \sigma_1'' + 2\sigma_1\sigma_1' = u' \quad , \qquad (9)$$

permit to verify that (8) into (7) leads to (2).
The following diagram illustrates our procedure:

$$\begin{array}{ccc} \Psi_1'' + (\lambda - u)\Psi = 0 & \xrightarrow{(5)} & \Phi'' + (\lambda - U)\Phi = 0 \;, \\ (3)\big\downarrow & & (7)\big\downarrow \qquad (10) \\ R' + R^2 + (\lambda - u) = 0, & \xrightarrow{(8)} & \tilde{R}' + \tilde{R}^2 + (\lambda - U) = 0 \;, \end{array}$$

Therefore, (8) represents the corresponding Darboux transformation for the Riccati equation in its normal form.

3. CONCLUSION

The expression (8) exhibits that an arbitrary Darboux mapping adds an exact derivative to the Riccati solution, which may be a underlying reason for the invariance of energy spectrum in the Schrödinger equation, being Ln $(R - \sigma_1)$ a generating function for Darboux transform in the Riccati scheme.

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