Higher Dimensional Differential Equations for Some Real World Simulation Processes and Solutions Thereof

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Abstract

Higher dimensional differential equations may express several real world simulation processes which depend upon their pre-history and subject to short-time disturbances. Such processes occur in the theory of optimal control, population dynamics, biotechnologies, economics, mathematical physics etc. So, the study of this class of dynamical systems is gradually gaining momentum. In the present work Avery-Peterson theorem has been envisaged for getting the positive periodic solutions of the corresponding differential equations for cones with impulses on time scales. By using the multiple fixed-point theorems we have shown through different lemmas and manipulation of several functions how the necessary criteria can be mathematically arrived at so that the results come to be feasible as well as effective.

1. INTRODUCTION

Recently, based on a fixed-point theorem in cones, Li *et al* [1, 2] investigated the periodicity of the following scalar system :

$y'(t) = -a(t)y(t) + g(t, y(t-\tau(t)));\quad t\neq t_j,\quad j\in Z,$

$y(t_j+)=y(tj-)+l_j(y(t_j))$, …………………… (i)

where $a\in C(R, (0, \infty))$, $\tau\in C(\mathbf{R}, \mathbf{R})$, $g\in C(\mathbf{R}\times[0, \infty), [0, \infty))$, $l_j\in C([0, \infty), [0, \infty))$, $j\in \mathbf{Z}$ and $a(t)$ & $\tau(t)$ are ω-periodic functions and $g(t, y)$ is ω-periodic with respect to its first argument. It is well known that system (i) includes many mathematical ecological models.

Also by using Krasnoselsii's fixed-point theorem and upper & lower solutions method Zhu and Li [3] found some sets of positive values λ determining that there exist positive T-periodic solutions to the higher dimensional differential equation of the form :

$x(n+1) = A(n)x(n) + \lambda h(n)f(x(n-\tau(n)));\quad n\in \mathbf{Z},$

where $A(n)=diag\ [a_1(n), a_2(n), a_3(n), \ldots, a_m(n)]$, $h(n)=diag\ [h_1(n), h_2(n), h_3(n), \ldots, h_m(n)]$, $a_j, h_j:\mathbf{Z}\to\mathbf{R}+$, $\tau:\mathbf{Z}\to\mathbf{Z}$ are T-periodic, $j=1, 2, 3, \ldots, m$, $t\geq1$, $\lambda>0$, $x:\mathbf{Z}\to\mathbf{R}_m$, $f:\mathbf{R}+m\to\mathbf{R}+m$ and $\mathbf{R}+m=\{(x_1, x_2, x_3, \ldots, x_m)T\in \mathbf{R}_m, x_j\geq0, j=1, 2, 3, \ldots, m\}$.

Motivited by the above, in this paper, we consider the following system :

$x\Delta(t) = A(t)x(t) + f(t, xt);\quad t\neq t_j,\quad j\in \mathbf{Z},\quad t\in \mathbf{T},$

$x(t_j+)=x(t_j-)+l_j(x(t_j))$, (ii)

where $\mathbf{T}$ is a ω-periodic time scale, $A(t)=(a_{ij}(t))n\times n$, $t\in \mathbf{T}$, is a non-singular matrix with continuous real-valued functions as its elements and $A(t+\omega)=A(t)$. $f=(f_1, f_2, f_3, \ldots, f_n)T$ is a function defined on $\mathbf{T}\times C(\mathbf{T}, \mathbf{R}_n)\to\mathbf{R}_n$ satisfying $f(t+\omega, xt+\omega)=f(t, xt)$ for all $t\in \mathbf{T}$ having $xt\in C(\mathbf{T}, \mathbf{R}_n)$ and $xt(s)=x(t+s)$ for all $s\in \mathbf{T}$, $x(t_j+)$ and $x(t_j-)$ representing the right and the left limits of $x(t_j)$ in the sense of time scales. In addition, if t_j is right-scattered, then $x(t_j+)=x(t_j)$ whereas if t_j is left-scattered, $x(t_j-)=x(tj)$; $l_j = (l_{j_1}, l_{j_2}, l_{j_3}, \ldots, l_{j_n})\ T\in C(\mathbf{R}_n, \mathbf{R}_n)$, $j\in \mathbf{Z}$. We assume that there exists a positive integer 'p' such that $t_j+p=t_j+\omega$ and $l_j+p=l_j$ where $j\in \mathbf{Z}$. For each interval l of $\mathbf{R}$ we denote $l\mathbf{T}=l\cap\mathbf{T}$. Without loss of generality we also assume that $[0, \omega)\mathbf{T}\cap\{t_j, j\in \mathbf{Z}\}=\{t_1, t_2, t_3, \ldots, t_p\}$.

Our aim is to use the Avery-Peterson theorem [4] for cones to establish the necessary cr iteria for having the positive periodic solutions of (ii). In the present article for each $x=(x_1, x_2, x_3, \ldots, x_n)T\in C([0, \omega]\mathbf{T}, \mathbf{R}_n)$ the norm of 'x' is defined as $|x|=\sup t\in [0, \omega]\mathbf{T}|x(t)|0$, where $|x(t)|0=\Sigma i=1n|x_i(t)|$, and when it comes to that x(t) is continuous, delta derivative, delta integrable and so forth, we mean that each element x_i is continuous, delta derivative, delta integrable and so forth.

In Section 2 we introduce some notations and definitions and state some preliminary results needed in the next section. In Section 3 we have established our main results for positive periodic solutions. Section 4 provides a brief conclusion.

2. Basic Formalism

Let $\mathbf{T}$ be a non-empty closed subset (time scale) of $\mathbf{R}$. The forward and backward jump operators $\sigma, \rho:\mathbf{T}\to\mathbf{T}$ and the graininess $\mu:\mathbf{T}\to\mathbf{R}+$ are defined respectively by $\sigma(t) = \inf\{s\in\mathbf{T}:s>t\}$; $\rho(t) = \sup\{s\in\mathbf{T}):s<t\}$; $\mu(t) = \sigma(t)t$. A point $t\in\mathbf{T}$ is called left-dense if $t>\inf\mathbf{T}$ & $\rho(t)=t$, right-dense if $t<\sup\mathbf{T}$ & $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. If $\mathbf{T}$ has a left-scattered maximum 'm', then $\mathbf{T}_k=\mathbf{T}\backslash\{m\}$; otherwise $\mathbf{T}_k=\mathbf{T}$. If $\mathbf{T}$ has a right-scattered minimum 'm', $\mathbf{T}_k=\mathbf{T}\backslash\{m\}$; otherwise $\mathbf{T}_k=\mathbf{T}$.

A function $f:\mathbf{T}\to\mathbf{R}$ is right-dense continuous provided that it is continuous at right-dense point in $\mathbf{T}$ and its left-side limits exist at left-dense points in $\mathbf{T}$. If 'f' is continuous at each right-dense as well as left-dense point, then it is said to be a continuous function of $\mathbf{T}$. The set of continuous functions $f:\mathbf{T}\to\mathbf{R}$ will be denoted by $C(\mathbf{T})=C(\mathbf{T}, \mathbf{R})$.

For $y:\mathbf{T}\to\mathbf{R}$ and $t\in\mathbf{T}_k$ we define the delta derivative of $y(t)$, $y\Delta(t)$, to be the number, if it exists, with the property that for a given $\varepsilon>0$ there exists a neighbourhood U of 't' such that $|[y(\sigma(t))-y(s)]-y\Delta(t)[\sigma(t)-s]|<\varepsilon|\sigma(t)-s|$ for all $s\in U$. If 'y' is continuous, then it is right-dense continuous and if it is delta differentiable at 't', then it is continuous at 't'.

Let 'y' be right-dense continuous. If $y\Delta(t)=y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

We say that a time scale $\mathbf{T}$ is periodic if there exists $p>0$ such that if $t\in\mathbf{T}$, then $t\pm p\in\mathbf{T}$. For $\mathbf{T}\neq\mathbf{R}$, the smallest positive 'p' is called the period of time scale.

Let $\mathbf{T}\in\mathbf{R}$ be a periodic time scale with period 'p'. We say that the function $f:\mathbf{T}\to\mathbf{R}$ is periodic with period ω if there exists a natural number 'n' such that $\omega=np$ and $f(t+\omega)=f(t)$ for all $t\in\mathbf{T}$. If $\mathbf{T}$ is ω-periodic, then $\sigma(t+\omega)=\sigma(t)+\omega$ and $\mu(t)$ is a ω-periodic function.

A n×n matrix-valued function A on a time scale $\mathbf{T}$ is defined as regressive with respect to $\mathbf{T}$ provided that $l+\mu(t)A(t)$ is invertible for all $t\in\mathbf{T}_k$.

Let $t_o\in T$ and A is a regressive n×n matrix-valued function. The unique matrix-valued solution

$Y\Delta = A(t)Y;\quad Y(t_o)=l,$

where 'l' denotes as usual the n×n identity matrix, is called the matrix exponential function at t_o and denoted by $eA(\cdot, t_o)$.

Now, let us introduce the lemma : if A is a regressive n×n matrix-valued function of $\mathbf{T}$, then $e_0(t, s)\equiv l$; $eA(t, t)\equiv l$; $ea(\sigma(t), s)=(l+\mu(t)A(t))eA(t, s)$; $eA(t, s)=eA-l(s, t)$ and $eA(t, s)eA(s, r)=eA(t, r)$.

We consider another lemma : let A be a regressive n×n matrix-valued function of $\mathbf{T}$ and suppose $f:\mathbf{T}\to\mathbf{R}_n$ is continuous and $t_o\in T$; then

$y\Delta = A(t)y + f(t);\quad y(t_o)=y_0,$

has a unique solution $y:\mathbf{T}\to\mathbf{R}_n$ given by

$$y(t) = eA(t, t_o)y_o + \int_{t_0}^{t} eA(t, \sigma(\tau))\, f(\tau)\Delta\tau.$$

From the above it follows that $f(t, x_t)$ is a continuous function of 't' for each $x\in C(\mathbf{T}, \mathbf{R}_n)$; for any $L>0$ and $\varepsilon>0$ there exists $\delta>0$ such that $\{x, y\in C(\mathbf{T}, \mathbf{R}_n), |x|\le L, |y|\le L, |x-y|\le\delta\}$ implying $|f(t, xt)-f(t, yt)|0<\varepsilon$ & $\forall t\epsilon[0, \omega]\mathbf{T}$ and the coefficient matrix A is a regressive n×n matrix-valued function of $\mathbf{T}$.

Let X be a Banach space and K a closed non-empty subset of X. K is defined as a cone if $\alpha u+\beta v\in K$ for all $u, v\in K$ and all $\alpha, \beta\ge 0$; $u, -u\in K$ implying $u=0$.

We define $K_r=\{x\in K| |x|\le r\}$. Let $\alpha(x)$ denotes the positive continuous concave functional on K *i.e.* $\alpha:K\to[0, +\infty)$ is continuous satisfying

$\alpha(\lambda x + (1-\lambda)y) \ge \lambda\alpha(x) + (1-\lambda)\alpha(y);\quad x, y\in K,\quad 0<1.$

Then we donate set $K(\alpha, a, b)=\{x|x\in k, a\le\alpha(x), |x|\le b\}$.

Let γ and θ are non-negative continuous convex functionals on K. Consider α to be a non-negative continuous concave functional on K and ψ a non-negative continuous functional on K. Then for positive real numbers a, b, c and d we define the following convex sets :

$K(\gamma, d) = \{x \in K | \gamma(x) < d\}$,

$K(\gamma, \alpha, b, d) = \{x \in K | b \le \alpha(x), \gamma(x) \le d\}$,

$K(\gamma, \theta, \alpha, b, c, d) = \{x \in K | b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\}$

and a closed set $R(\gamma, \psi, a, d) = \{x \in K | a \le \psi(x), \gamma(x) < d\}$.

Avery-Peterson fixed-point theorem is important for getting the main result : let γ and θ are non-negative continuous convex functionals on K, α a non-negative continuous concave functional on K and ψ a non-negative continuous functional on K satisfying $\psi(\rho x) \le \rho\psi(x)$ for $0 \le \rho \le 1$ such that for some positive numbers E and 'd', $\alpha(x) \le \psi(x)$ and $|x| \le E\gamma(x)^*$ for all $x \in K(\gamma, d)^-$. Suppose that $H: K(\gamma, d)^- \to K(\gamma, d)^-$ is completely continuous and there exist positive numbers a, b and c with $a<b$ such that $\{x \in K(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \neq \emptyset$ & $\alpha(Hx) > b$ for $x \in K(\gamma, \theta, \alpha, b, c, d)$; $\alpha(Hx) > b$ for $x \in K(\gamma, \alpha, b, d)$ with $\theta(Hx) > c$ and $0 \notin R(\gamma, \psi, a, d)$ & $\psi(Hx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then H has at least three fixed points $x_1, x_2, x_3 \in K(\gamma, d)^-$ such that $\gamma(x_i) \le d$ for i=1, 2, 3 giving

$b < \alpha(x_1)$, $a < \psi(x_2)$, $\alpha(x_2) < b$, $\psi(x_3) < a$.

In order to obtain the existence of periodic solutions of system (ii) we make the following preparations. Let us define $PC(\mathbf{T}) = \{x = x_1, x_2, x_3, ---, x_n\} T: \mathbf{T} \to \mathbf{R}_n |x_i| [t_j, t_j+1) \mathbf{T} \in C((t_j, t_j+1)\mathbf{T}, \mathbf{R})$, □$x(t_j-) = x(t_j) x(t_j+)$, $j \in \mathbf{Z}$, i=1, 2, 3, ---, n}.

Now, set

$X = \{x(t) : x(t) \in PC(\mathbf{T}), x(t+\omega) = x(t)\}$

with the norm defined by $|x| = \sup t \in [0, \omega] \mathbf{T} |x(t)|0$, where $|x(t)|0 = \Sigma i=1n |x_i(t)|$. Then X is a Banach space.

For convenience we introduce the following notations :

$G(t, s) eA(\sigma(s), s) = [eA(0, \omega) - l] - 1 eA(t, \sigma(s)) eA(\sigma(s), s) = [eA(0, \omega) - l] - 1 eA(t, s) = E(t, s) := (E_{ik}) n \times n$ for $t, s \in \mathbf{T}$, i, k=1, 2, 3, ---, n;

$A_0 := \min 1 \le i, k \le n \inf s, t \in [0, \omega] \mathbf{T} |G_{ik}(t, s)|$;

$B_0 := \max 1 \le i, k \le n \sup s, t \in [0, \omega] \mathbf{T} |G_{ik}(t, s)|$;

$A_1 := \min 1 \le i, k \le n \inf s, t \in [0, \omega] \mathbf{T} |E_{ik}(t, s)|$;

$B_1 := \max 1 \le i, k \le n \sup s, t \in [0, \omega] \mathbf{T} |E_{ik}(t, s)|$;

$A_2 := \min \{A_0, A_1\}$;

$B_2 := \max \{B_0, B_1\}$;

$A_3 := \min 1 \le k \le n \inf s, t \in [0, \omega] \mathbf{T} |\Sigma i=1n G_{ik}(t, s)|$;

$B_3 := \max 1 \le k \le n \sup s, t \in [0, \omega] \mathbf{T} |\Sigma i=1n G_{ik}(t, s)|$;

$A_4 := \min 1 \le k \le n \inf s, t \in [0, \omega] \mathbf{T} |\Sigma i=1n E_{ik}(t, s)|$;

$B_4 := \max 1 \le k \le n \sup s, t \in [0, \omega] \mathbf{T} |\Sigma i=1n E_{ik}(t, s)|$;

$A_5 := \min \{A_3, A_4\}$;

$B_5 := \max \{B_3, B_4\}$.

Hereafter we assume that $A_i > 0$ & $B_1 > 0$ for i=0, 1, 2, ---, 5 and $G_{ik} f_k > 0$ & $E_{ik} l_{jk} > 0$ for all i, k=1, 2, 3, ---, n & $j \in \mathbf{Z}$. Let $K = \{x = (x_1, x_2, x_3, ---, x_n) T \in X : x_i \ge \delta |x_i|, t \in [0, \omega] \mathbf{T}$, i=1, 2, 3, ---, n}, where $\delta = A_2/B_2 \in (0, 1)$ and A_2 & B_2 are as defined above. Obviously, K is a cone in X.

We claim that

$eA(\sigma(s+\omega), t+\omega) = eA(\sigma(s), t)$.

In fact $eA(\sigma(s+\omega), t+\omega) = eA(\sigma(s)+\omega, t+\omega) = A(\sigma(s), t)$. Similarly, we can get $eA(t+\omega, \sigma(s+\omega)) = eA(t, \sigma(s))$ implying $G(t+\omega, s+\omega) = G(t, s)$.

Now, define a mapping H by

$(Hx)(t) = \int_t^{t+\omega} G(t, s) f(s, x_s) \Delta s + \Sigma j : t_j \in [t, t+\omega) \mathbf{T} G(t, t_j)\, eA(\sigma(t_j), t_j) l_j(x(t_j))$,

$\Rightarrow (Hx)(t) = \int_t^{t+\omega} G(t, s) f(s, x_s) \Delta s + \Sigma j : t_j \in [t, t+\omega) \mathbf{T} E(t, t_j)\, l_j(x(t_j))$

for all $x \in K$ and $t \in \mathbf{T}$. Then,

$(Hx)(t) = ((H_1 x)(t), (H_2 x)(t), (H_3 x)(t), ---, (H_n x)(t)) T$,

where $(H_i x)(t) = \int_t^{t+\omega} \Sigma k = 1n G_{ik} f_k(s, x_s) \Delta s + \Sigma j : t_j \in [t, t+\omega) \mathbf{T} \Sigma k=1n E_{ik} l_{jk}(x(t_j))$.

3. Result

We, now, fix $\eta, l \in [0, \omega] \mathbf{T}$, $\eta \le l$, and let the non-negative continuous concave functional α, the non-negative continuous concave function ψ and the non-negative continuous functionals γ and θ are defined on the cone K by

$\alpha(x) = \inf t \in [\eta, l] \mathbf{T} |x(t)|0$,

$\psi(x) = \theta(x) = \sup t \in [0, \omega] \mathbf{T} |x(t)|0$,

$\gamma(x) = \sup t [0, \omega] \mathbf{T} |(\phi x)(t)|0$

respectively, where $(\phi x)(t)=\int_0^\omega h(t, s)x(s)\Delta s$ and $h(t, s)\in C(\mathbf{T}_2, \mathbf{R})$.

The functionals defined above satisfy the following relations :
$\alpha(x) \le \psi(x) = \theta(x); \forall x\in K$.

We recall the lemma : for $x\in K$ there exists a constant $E>0$ such that sup $t\in [0, \omega]\mathbf{T}|x(t)|0\le E$ sup $t\in [0, \omega]\mathbf{T}|(\phi x)(t)|0$.
Moreover, for each $x\in K$,
$|x|=\sup t\in [0, \omega]\mathbf{T}|x(t)|0\le \sup t\in [0, \omega]\mathbf{T}|(\phi x)(t)|0L\delta=E\gamma(x)$
…………………… (iii)

We also find that $\psi(\rho x)=\rho\psi(x)$ for $\forall\rho\in [0, 1]\mathbf{T}$ for all $x\in K$. Therefore, by (iii) the condition (ii) of Avery-Peterson fixed-point theorem is satisfied.

Hence, the criteria for having the solutions of the higher dimensional differential equations for cones with impulses on time scales are that there exist constants a, b, d>0 with $a<b<b/\delta<d/L$ such that
(1) $|f(t, u)|0 < d/B_5L\omega - l_1M/\omega$ for $0\le|u|0\le Ed$, $t\in [0, \omega]\mathbf{T}$,
(2) $|f(t, u)|0 > b/A_5\omega - l_m/\omega$ for $b\le|u|0\le b/\delta$, $t\in [\eta, l]\mathbf{T}$ and
(3) $|f(t, u)|0 < a/B_5\omega - l_2M/\omega$ for $0\le|u|0\le a$, $t\in [0, \omega]\mathbf{T}$,

where $l_1M=\max 0\le|u|0\le Ed\Sigma j=1p|l_j(u)|0$,
$l_2M=\max 0\le|u|0\le a\Sigma j=1p|l_j(u)|0$
and $l_m=\min b\le|u|0\le b/\delta\Sigma j=1p|l_j(u)|0$.

4. Conclusion

By using a multiple fixed-point theorem *i.e.* Avery-Peterson theorem for cones some criteria are established for the existence of positive periodic solutions for a class of higher dimensional functional differential equations with impulses on time scales of the following form :
$x\Delta(t) = A(t)x(t) + f(t, xt);\quad t\neq t_j,\quad t\in \mathbf{T},\quad x(t_j+)=x(t+)+l_j(x(t_j))$,
where $A(t)=(a_{ij}(t))n\times n$ is a non-singular matrix with continuous real-valued functions as its elements. The yield thus obtained is meant for the feasibility and effectiveness of the results.

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