The Economic Journal of Nepal, Vol. 31, No. 3, July-September 2008 (Issue No. 123) © CEDECON-TU

Renewable Natural Resource Extraction: Theory, Empirics and an Extension of Hotelling's r-Percent Rule

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Abstract

This paper reviews Hotelling's work on the extraction of exhaustible natural resources, and it's extensions under different scenarios. The existing theory of exhaustible natural resources does not deal with the externality effect of the residuals from the extraction and processing activities, and predicts that neither the extraction path nor the user cost depends on the price path of the final product. By allowing residual emission with externality while extracting and processing the exhaustible natural resources we find that both the extraction path and the user cost depend on the price path of the final product.

Introduction

The ultimate exhaustion of exhaustible natural resources has been regarded as a serious economic problem. Hotelling (1931) develops the basic theory of exhaustible resources using calculus of variations that has been the central issues of the research agenda in the case of exhaustible resources since then. Several researchers provide extensions to the basic theory under different market structures, and under uncertainties of different kinds. In doing so, they use somewhat newer method of dynamic analysis called the optimal control theory.[1] Another line of research has been devoted to some form of empirical tests of the Hotelling's basic rule that the market price of the natural resources compounds at the market rate of interest. Hotelling's seminal paper formally shows that it is profitable to limit the current extraction of natural resources and conserve it for the future generations if scarcity is an issue.

Hotelling recognizes the fact that though the social optimum and the free market outcomes are identical under simplifying assumptions, uncertainty and unexpectedness in the mineral discoveries provide the ground for government regulations. He also shows that with constant costs, resource would last longer in case of monopoly than the case of competitive market

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[1] This new method provides the same result as the calculus of variations.

or social planning. Further development of the theory of exhaustible natural resources extraction has been summarized in a two- paper series by Chermak and Patrick (2001, 2002). These papers provide up-to-date development of the modeling practices in the field. The basic results of these and related papers (e.g., Slade, 1982; Farrow, 1985; Stollery, 1985; Young, 1992; Slade and Thille, 1995; Halvorsen and Smith, 1991; Krauthkraemer, 1998) are that neither the extraction path nor the user cost depends on the price path of the final product. The lack of sensitivity of the extraction path and the user cost to the market price is rather surprising as we generally expect that market price should play an important role for extraction activities of scarce resources in a decentralized economy. In these existing models, an implicit assumption is that there would be no externality effect of the residuals when extracted resources are processed for their use.

The objective of this paper is to review the extensions of the Hotelling's original work including some empirical and theoretical works by subsequent researchers and to develop an extension of the general theory of exhaustible resource extraction. The proposed extension shows that those earlier results are special cases of resource extraction when externality effect of the residuals is left-out. When we adopt the general formulation where waste generation is allowed in the production process and its externality is taken into account, we get the price dependent extraction path. The main contribution of this paper is the recognition of the externality effect of the production activities and the introduction of the damage function into the objective function so that the firm must take into account not only the private extraction cost, but also the social cost that is reflected into the damage function. The public bad nature of the pollutants that originate while processing the extracted natural resources for final product and its inclusion in the objective function makes the method of determining the optimal extraction path more general than the usual methods in which the externality aspects has been ignored.

Hotelling's Basic Model

The basic Hotelling model of exhaustible resources uses simplified assumptions that stock of the natural resource under consideration is known with given quality and the extraction technology does not change over time. With the fixed extraction cost (not related to the units of extraction), the basic Hotelling's model (Hotelling 1931) examines the intertemporal allocation of finite stock of known exhaustible resources. The firm chooses the time path of the extraction rate $q(t)$ that maximizes the discounted stream of the net benefits from extraction such that cumulative sum of the resource extraction over period T does not exceed the total resource availability (R_0). Formally, the objective is to maximize:

$$(1) \qquad \int_0^T e^{-rt} P(t)q(t)dt - F(0)$$

Subject to:

(2) $$\int_0^T q(t)dt \le R_0$$

which can be written as

(2a) $\dot{R} = -q_{(t)}$

Where $R(T) = 0$, $\dot{R}$ is the rate of change of the resource stock, r is the discount rate, $P(t)$ is the market price at period t (which is given under prefect competition), $F(0)$ is the fixed costs which is independent of the rate of extraction, and T is the terminal time where the resource stock will be exhausted completely. The Hamiltonian of the basic Hotelling Model is given by:

(3) $H = e^{-rt} P(t)q(t) - F(0) - \lambda(t)q(t)$

The first order conditions under perfectly competitive market are given by:

(4) $$\frac{\partial H}{\partial q} = e^{-rt} P(t) - \lambda(t) = 0$$

(5) $$\frac{\partial H}{\partial K} = -\dot{\lambda} = 0$$

Taking time derivatives of (4) and substituting $\dot{\lambda} = 0$ from (5), we get

(6) $$\frac{\dot{P}}{P} = r$$

which is Hotelling's well-known '*r-percent*' rule. This r-percent rule implies that at equilibrium, returns from alternative assets should be equal to the rate of ıeturns from the exhaustible resources. As the rate of interest is the returns from the alternative assets to holding exhaustible resource stock, the *in situ* value of exhaustible resource should appreciate at the market rate of interest. The implications of the basic Hotelling's model are that with a stationary demand curve, extraction decreases as the resource price increases over time with the rate of interest. Essentially, it is equivalent to say that market takes care of the resource scarcity (reacts with higher prices), as it is profitable to postpone the current extraction and conserve for the future. This is the central message of the Hotelling theory.

Hotelling's Rule Under Monopoly

In his original paper, Hotelling (1931) shows that the period of operation is finite under competition but infinite under monopoly. He asserts that it is also true even in the situation where the cumulative production affects the resource price. He uses a numerical example to show that the resource lasts longer in the hand of monopolist than the case of social planner.

Subsequent researchers (Stiglitz, 1976; Lewis, Matthews and Burness, 1979) developed detail analysis of the monopoly issue, in which the result is inconclusive. If demand curve is iso-elastic, and extraction cost is zero, then the monopolist's extraction path is given by:

$$(7) \quad \frac{\dot{q}}{q} = \frac{r - \frac{f'}{f}}{\alpha - 1}$$

This expression can be obtained by using the constant elasticity demand function given by the expression:

$$(8) \quad P(t) = f(t)q^{\alpha-1}$$

Equation (7) holds for both competitive market as well as the monopolist. This implies that at each time period prices are identical in the competitive and the monopoly markets and so must be the rate of extraction of the natural resource (Stiglitz, 1976: 657). This conclusion is based on the assumptions that the elasticity of demand$[1/(1-\alpha)]$ is constant and extraction cost is zero. If we assume an increasing elasticity of demand over time due to availability of substitutes for a given resources, the growth rate of resource price is given by the expression:

$$(9) \quad \frac{\dot{P}}{P} = r - \frac{\alpha'}{\alpha}$$

where $\alpha' > 0$ (which is true if elasticity of demand increases over time). Equation (9) is true for monopolist, and competitive market still has the price path given by (6) indicating that price increases at a slower rate under monopoly than the competitive market. Given the resource constraint, the implication of equation (9) is that monopolist takes the more conservationist extraction path. Stiglitz concludes that this conclusion does not change if unit extraction costs are incorporated in the model.

The generalization that monopolistic extraction rate is slower than the social optimal (or the competitive market) would be invalid under two assumptions: a) existence of quasi-fixed costs; or, b) demand elasticity increases with consumption instead of time (Lewis et al., 1979). If we assume that the resource owner incurs a cost that do not vary with the extraction rate (e.g. leasing fees, capital costs, maintenance fees), which is very likely, or if demand elasticity increases with consumption, then monopolist depletes resources faster than the competitive market case. In either case, resource remaining is always less than socially optimal and depletion occurs too soon in the case of monopoly. The reason for faster extraction under monopoly is that under the given assumptions of quasi fixed costs, and demand elasticity increasing with consumption, discounted marginal revenue at current period becomes higher than the future periods. Monopolist extracts more in current period so that the marginal revenues between periods become equal (Lewis, Matthews and Burness, 1979). The inference that can be drawn form above analysis is that Hotelling's claim that the monopolist is more conservationists, is indeterminate and it is a matter of empirical issue. As discussed above, the actual outcome depends on the type of assumptions we incorporate in the model.

Hotelling's Rule Under Uncertainty

The basic Hotelling model assumes that the stocks of the exhaustible resources are known with certainty. In fact, the stocks of the exhaustible resources are not known with certainty. Exploration and development activities are very common features of the mineral industries, in which final outcomes cannot be fully anticipated. In this section the effects of the demand and reserves uncertainties in the Hotelling's *r*-percent rule is considered in detail.

With constant extraction costs and risk neutral firms, Pindyck (1980) shows that neither demand nor reserve uncertainty affects the expected price dynamics in the competitive or monopolistic markets. In this case also Hotelling's *r*-percent rule appears to be robust. However, if extraction costs are not constant, the *r*-percent rule no longer applies even in the deterministic case. The distinctive feature of Pindyck paper is that it assumes that producers always know the current demand and reserves but do not know what demand and reserves will be in the future. In this case, stochastic processes drive the market demand function and the reserve level. So, the market demand is given by:

(10) $p=p(q,t) = y(t)f(q)$

with $f'(q) < 0$, and $y(t)$ a stochastic process where

$$\frac{dy}{y} = \alpha dt + \sigma_1 \varepsilon_1(t)\sqrt{dt} \tag{11}$$

where $\varepsilon_1^{(t)} \sim N(0,1)$ is serially uncorrelated normal random variable with 0 mean and variance 1. Equation (11) shows that uncertainty about demand grows continuously over time. Reserves are also assumed to fluctuate randomly over time given by the following stochastic process, similar to the demand case:

$$dR = -qdt + \sigma_2 \varepsilon_2(t)\sqrt{dt} \tag{12}$$

where q is the rate of extraction of resources. Initial reserve R_0 and current reserve $R(t)$ are known exactly, but expected reserve R_e is not known once the production process starts. The effective reserve is a random variable with mean R_0, and variance $\sigma^2{}_2 T$, given by the following expression:

$$R_e = R_0 + \sigma_2 \int_0^T \varepsilon_2(t)\sqrt{dt} \tag{13}$$

So, the firm's problem is to choose $q(t)$ over time to optimize the following expected value:

$$\text{(14) } \operatorname{Max}_q E_0 \int_0^T [y(t)f(q) - C_1(R)]qe^{-rt}dt$$

Using eq. (14) we get optimal condition that the shadow price of the resource equals the incremental profit that could be obtained by selling additional unit (Pindyck, 1978). The expected rate of change of price is given by the following equation:

$$(15) \qquad E_t\left(\frac{dp}{dt}\right) = r[p - C_1(R)] + \frac{1}{2}\sigma_2^2 C_1^{"}(R)$$

The important point to note here is that if the production cost is linear, equation (15) reduces to the certainty case. Also, if the average production cost $C_1(R)$ is constant, we will get the Hotelling's *r-percent* rule even in the case of uncertain future demand and the future reserves. The deviation from the certainty case occurs in the case that $C_1(R)$ is nonlinear. These two extensions show that Hotelling's insight was quite robust.

Hotelling's Rule Under Empirical Test

The theory of the exhaustible resources has been tested under different theoretical backgrounds, using different levels of aggregation, different time periods, and different industry groups. Initial tests was done by Barnett and Morse (1963), where they looked at the relative-price trends for five groups of natural resource commodities and found that real prices had fallen over time (1870-1957), meaning that scarcity was not a problem. They hypothesized that if the scarcity is a problem, then the unit price of the resource extracted should have increasing trend over time, not declining. This type of test in which price path is tested explicitly as in the case of Barnett and Morse (1963) have been changed in subsequent studies where the tests are done either optimizing the dynamic profit functions, or minimizing the dual restricted cost function first and using that indirect cost function to optimize the profit function. In the empirical studies, the type of tests used to test the Hotelling's principle can be summarized in the following forms:

$$(16) \qquad \frac{\dot{P}}{P} = r$$

$$(17) \qquad \lambda = e^{-rt}(P - C_q)$$

$$(18) \qquad \lambda = -e^{-rt}\,\bar{C}_q$$

Equation (16) is the basic Hotelling's *r*-percent rule that says the price of the exhaustible resource will rise at the rate of interest at equilibrium. This is true when marginal extraction cost is zero. As discussed above, this type of test is done in Barnett and Morse (1963). When non-zero marginal extraction cost (*Cq*) is included in the basic model, the test becomes shadow price test, given by equation (17), which states that the shadow price (ë) rises at the rate of interest ($\frac{\dot{\lambda}}{\lambda} = r$). When we account for the effects of reserve remaining in the extraction cost, and if we use the indirect cost function for the optimization, then the test is given by equation (18). Equation (18) comes from different formulations of the basic Hotelling model. There are two basic differences between the first two predictions [16 and 17] and the last one (18): whether the extracted product is final good or it requires processing in order to make it final product; and whether the cost function is direct or indirect. In equation (17) the shadow price is given by ($P-C_q$). This approach was used in Farrow (1985), and Miller and Upton (1985). In equation (18) the shadow price is measure by - $\bar{C}_q$, which represents the indirect marginal cost of extraction (Halvorsen and Smith 1991; Chermak and Pratick 2001, 2002). Therefore, the test methodologies of the theory of exhaustible resource can be put into the following three categories:

a) *Direct test for the Hotelling's r-percent rule*: In this case firm is modeled as producing the exhaustible resource with production costs independent of stock and then testing whether price increases at the rate of interest (Barnett and Morse1963).

b) *Explicit shadow price path test*: This approach derives the dynamic optimality conditions at the firm level which extracts resources, and uses those conditions to test if the estimated parameters are consistent to the theoretical prediction (Farrow 1985, Miller and Upton 1985, Stollery 1985, Slade and Thille 1997).

c) *Implicit shadow price path test*: This approach allows processing of the resources such that there is a difference between gross and the final product. It is different from (b) in the sense that it estimates the indirect cost function restricted by the dynamic optimality condition and tests whether the parameters of the restricted cost function are consistent with those of the unrestricted cost function (Halvorsen and Smith 1991, Chermak and Patrick 2001, 2002).

Miller and Upton (1985) use a slightly different method to test the equation (17) type of test. They use the discrete time model in which they argued that in a world where output prices, net of extraction costs, obey the Hotelling's principle, the value of the total reserves in any mineral property depends solely on the current spot price per unit of the mineral, net of current extraction costs. This is called the Hotelling's Valuation Principle, a corollary to the *r-percent* rule. Mathematically, it can be written as:[2]

(19) $V_0 = (P_0 - C_0) R_0$

Where V_0 is the net present value of the profit from extraction, P_0 is the current market price of the extracted resources, C_0 is the marginal extraction cost, and R_0 is the total reserves available initially. This variant of Hotelling's Valuation Principle is tested using following econometric model:

$$(20) \qquad \frac{V_0^{it}}{R_0^{it}} = \alpha + \beta(P_0^{it} - C_0^{it})$$

This test is basically called the shadow price test where *t* refers to time period, *i* refers to the company (firm), *0* refers to the current value as of the sample date *t*. The dependent variable is the observed market values per unit of reserves for mineral properties of given companies at a given point in time. The explanatory variable is the current output price net

[2] It is derived from maximization of the discounted present value of the profits subject to the resource constraint. This gives Hotelling's r-percent rule in discrete time. Substituting the r-percent rule in the objective function assuming constant returns to scale, we get equation (19).

of marginal extraction costs. Here the test for Hotelling's Principle implies: $\alpha = 0$, and $\beta = 1$ under the constant returns to scale assumption.

Using monthly data (12/1979-8/1981) from stock market valuations of the oil and gas reserves of a sample of 39 US oil and gas producing companies,[3] Miller and Upton (1985) found that the â coefficient is not different from 1, indicating that the alternative Valuation Principle supports the Hotelling's basic principle. They used stock market valuations of the oil and gas reserves of a sample of U.S. firms for their analysis. To test the dynamic efficiency of the extracting firm, Farrow (1985) estimates the following equation:

$$(21) \qquad \Delta m(t) = \delta m(t\text{-}1) + \beta_1 C_x(t) + u_t$$

This equation is the discrete time counter part of the continuous time necessary condition for profit maximization, and comes from the formal discounted profit-maximization problem of the competitive mining firm.[4] Here $m(t)$ is the current value shadow price, $C_x(t)$ is the marginal extraction cost of the resource remaining in the ground, and u_t is the stochastic error term. The null hypotheses for the dynamic efficiency condition are: ä = 0 and â = 1. The claim is that Hotelling model is consistent with the data if these hypotheses are not rejected.

Using time series data of a competitive U.S. hard rock mining firms for the period January 1975 to December 1981, Farrow did not find the support for the theory and maintained hypothesis as the discount rate (ä) is significantly negative. He used the translog cost function for estimation purpose.

Halvorsen and Smith (1991) use indirect or restricted cost function (in the generalized Cobb-Douglas format), dual to the production function, for final output to test the theory of exhaustible resources. Their dynamic optimality condition is given by

$$(22) \qquad \mu = r\mu - \partial C / \partial Z$$

where Z is gross production (cumulative extraction), r is the market rate of interest, and $\grave{\imath}$ is the shadow value of the marginal unit extracted (difference between value of the marginal product and marginal extraction cost, or the current value user cost). As data on the shadow price (user cost) and the effect of the cumulative extraction on marginal cost are not observed, they test the coefficients in the restricted cost function, and Hausman specification test is used to test the null hypothesis that the dynamic optimality condition is satisfied through the parameter constraint of the indirect cost function. For that they estimate system of equations consisting: i) restricted cost function, ii) cost share equation of reproducible inputs, and iii) average restricted cost equation of cumulative extraction. As a proxy of the shadow price, they use $-C_z$, negative of the marginal cost of extraction in terms of cumulative extraction.

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[3] The total sample observations were 94, in which 19 companies had three reserve evaluations, and 17 companies had two reserve evaluations.

[4] The firm's problem is to maximize $\int e^{-\delta t}\{P(t)q(t)-C[q(t), X(t), W(t)]\}dt$ subject to q(t), where X(t) is stock remaining, W(t) is a vector of input prices, and rest are same as in the basic Hotelling model.

Using annual time series data for a Canadian metal mining industry for the period 1954-1974, the dynamic optimality condition is tested under various discount rates (both constant, and time varying) in which the null hypothesis is rejected under different (2-20%) discount rates. The authors agree that this rejection might be due to the higher level of data aggregation. Chermak and Patrick (2001) extend the Halvorsen and Smith (1991) type of test for the dynamic consistency of the extraction of natural gas resources. They use the following equation to test the dynamic consistency of the firms' behavior.

$$(23) \quad \dot{\mu} = r\mu + \partial C / \partial R$$

The difference between eq (22) and eq (23) is that in the later equation, the stock remaining (R) is used instead of cumulative extraction (q, which was denoted by Z in Halvorsen and Smith (1991)) to get the marginal cost function. In this paper distinction is made between the gross product (q) and the final product (Z). Three different types of indirect cost functions (unrestricted, restricted, and final average) in the form of a Generalized Cobb-Douglas format are estimated using monthly data between 1987-1991 from 29 tight sand gas wells in the U.S. (443 observations). The following equation is derived from eq (23) using the indirect cost function, which is used to restrict the parameters of the indirect final cost function.

$$(24) \quad e^{rt}[\bar{C}_q |_{t=0}] - \bar{C}_q = \sum_{x=0}^{t} (1+r)^{t-x} \bar{C}_R$$

The hypothesis tested is if (24) is satisfied by parameter estimates for the following indirect cost function using Hausman specification test:

$$(25) \quad \bar{C} = \bar{C}(W, z, q, R, t, ...)$$

where is the indirect cost function coming form cost minimization problem given the production function, W is the vector of input prices, q is the gross production, z is the final product, R is the resource stock, and t is the time variable. The final cost function for econometric estimation is specified as follows:

$$(26) \quad \ln \bar{C} = \beta_q \ln q + \beta_z \ln z + \beta_R \ln R + ... + e$$

Here the omitted part of the equation refers to the production months, firm's ID and years of production. For the restricted cost function, the following restrictions are tested using Wald statistic:

$$(27) \quad H_0: \hat{a}_z = -\hat{a}_q \; vs. \; H_1: \hat{a}_z \text{ `" } -\hat{a}_q$$

The statistical results support the null meaning that the restricted cost function is econometrically appropriate for analyzing the firm's behavior. Using the Hausman statistics (estimating (25) with and without (24)) with range of discount rates (2-20%), Chermak and Patrick (2001) found that firms are behaving in a manner consistent with Hotelling's theory.

Another paper by Chermak and Patrick (2002) builds on the same data and the theory developed in Chermak and Patrick (2001). The distinctive feature of this paper is that it synthesizes four different published studies (Farrow (1985), Halvorsen and Smith (1991), Slade and Thille (1997), and Chermak and Patrick (2001)) using the same data set used in Chermak and Patrick (2001). Originally, these studies are different in terms of problem formulation, resources considered, level of aggregation, data and time period, measurement of user costs, and the method of testing. Using new data set, original Farrow (1985) results do not change (no support to theory). Original Halvorsen and Smith (1991) results did not find support to the theory, but the results from new data set provide support to the theory. Slade and Thille (1997) found support to the theory, but new data did not support it. These results from the unified data set are again mixed, and no definite conclusion can be drawn from these new results about the theory. Empirical test results of six different studies are summarized in the following table. One striking feature of the empirical tests reviewed here is that all tests used time series data, ranging from monthly to annual (table 1). However, none of the tests explicitly analyzed the time series data issues like non-stationarity (unit root), and co-integration, for example.

Table 1 Summary of Empirical Tests and Results of Hotelling's Principle

Researchers (Year)	Nature of data (Period)	Type of test	Type of Product	Type of Market	Level of aggregation	Test Result
1. Barnett & Morse (1963)	Annual (1870-1957)	Direct price path	Different category	Competitive	Industry	Reject
2. Miller & Upton (1985)	Monthly (12/79-8/81)	Shadow Price	Oil & Gas	Competitive	Firms	Do not reject
3. Farrow (1985)	Monthly (1/75-12/81)	Shadow Price	Metals	Competitive	Single firm	Reject
4. Halvorsen & Smith (1991)	Annual (1954-74)	Shadow Price(Implicit)	Metals	Competitive	Industry	Reject
5. Chermak & Patrick (2001)	Monthly (5/87-6/91)	Shadow Price (Implicit)	Gas	Competitive Wells	Individual	Do no reject
6. Chermak & Patrick (2002)	Monthly (5/87-6/91)	Replication of: 3, 4, & 5	Gas	Competitive	Individual Wells	3: Reject, 4 & 5: Don't reject

Using time series for econometric analysis requires the tests for these time series data issues (Hamilton 1994). If data are non-stationary or co-integrated, then these issues must be addressed before using those data for analysis. This might be one of the reasons why empirical tests in Chermak and Patrick (2002) about Hotelling's principle gave divergent results even though uniform data set was used to re-evaluate four previous studies. It is very likely that the test results may differ once time series data issues are taken into account. As those data were proprietary, it is not possible to put those data into closer scrutiny.

An interesting feature of Chermak and Patrick (2001, and 2002) papers is to differentiate between gross product (q) and the final product (z) These two papers are obviously very clear and consistent for technical details, and builds in the foundation of previous models.

The technical aspects of these papers are summarized below as those provide the background of extension that is presented in section VI. The firm's indirect cost function for the final product is:

(28) $(z(t), q(t), W, R(t), t)$ a" $\min_x W.X$, s.t. $z(t) = f(X^f, q, t)$ and $q = g(X^g, R, t)$

where W is input price vector, $R(t)$ reserve remaining at time t, $q(t)$ gross production, $z(t)$ final production where, z d" q, and X is the input vector.The competitive firm's objective is

(29) $$\text{Max } \pi_{[z,q,T]} = \int_0^T e^{-rt}[P(t)z(t) - \bar{C}(z(t), q(t), W, R(t), t)]dt$$

subject to:

(30) $$\dot{R}(t) = -q(t), R(0) = \bar{R}, R(t) \geq 0,$$

where P is given market price of final product, r is discount rate, is change in the resource stock, $R(0)$ is the total known reserves, and $R(T)$ is the remaining reserve at the terminal time, T. Using the optimal control theory for optimization, Chermak and Patrick (2002) find the following two different extraction paths of the resources:

(30 a) $$\dot{q} = \frac{r\bar{C}_q - \bar{C}_{qz}\dot{z} - \bar{C}_{qR}\dot{R} - \bar{C}_{qt} - \bar{C}_R}{\bar{C}_{qq}}$$

(30 b) $$\dot{q} = \frac{rC_q + \dot{P} - rP - C_{qR}\dot{R} - C_{qt} - C_R}{C_{qq}}$$

In (30 a) the optimal extraction path is free from the final product price path (), where as in (30 b) the optimal extraction path depends on the final product price path. Result in (30 a) depends on the assumption that there is a difference between final product (z) and the gross extraction (q), i.e. $z < q$. If we assume that there is no difference between z and q, then we get (30 b). In the next section, we show that these findings change once we recognize that the difference between z and q becomes a stock pollutants, and the firm has to take care of the pollution aspects of the extraction activities by incorporating the damages due to pollution in the decision making process.

An Extension of the Exhaustible Resource Models

This section presents an extension of the theory of exhaustible resources under the assumption that gross extraction differs from the final product, and the difference becomes the stock pollutants. The stock pollutants are bad as they result into degradation of the environment and hence produce the disutility to individuals and to the society. For example, consider the cement production process. It involves extraction of ore (limestone), processing it, and get the final product (cement). During the process, it produces residuals, smoke and dust particles that pollute the air and the surrounding environment. Polluted air leads to health damage, and other unwanted results, while the polluted environment might be responsible to the reduced level of output in the agricultural activities, or reduction in the property values in the surrounding areas. Another example is extraction, transportation, and conversion of coal into electricity. In this process, the residual goes to the environment in the form of stock pollutants.

This section proposes a theoretical extension of the exhaustible resources models to analyze the optimal extraction at the firm level when there is externality involved with it. Basically, it analyzes the optimal extraction path at the firm level where the externalities are taxed appropriately (market based solution)[5] by the regulatory body in the presence of a stock pollutant.

Consider a firm that extracts an exhaustible natural resource at the rate $q(t)$. Suppose that there is a distinction between the extracted materials (q), and the final products (z) as in Chermak and Patrick (2001, 2002) so that the firm needs to process the extracted resource in order to make it a final product. We assume that this conversion of extracted resources produces wastes, $W(t)$, that damages the surrounding environments (say air or water, for example). We assume that if the firm chooses not to process the extracted resources into the final product, then there would be no stock pollutants. Assume also that the market for the final product is competitive. For the modeling purpose we use the same nomenclature as in Chermak and Patrick (2001, 2002,) and we have following two additional variables: $W(t)$, Accumulated waste, and $D(W)$, Damage due to the waste $W(t)$. From the firm's perspective, we can proceed either through profit maximization approach, or the cost minimization approach. Here we use the latter approach following Chermak and Partick[6] (2001, 2002), and Halvorsen and Smith (1991) in order to make comparison between the optimal extraction paths. In this case, we get the indirect cost function for the final product by minimizing the cost equation subject to the production technology (both extraction, and processing).

Here the firm not only has to take care of the production costs, but also the damages associated with the stock pollutant. It is assumed that the regulatory body exactly knows the damages caused by the stock pollutants and puts the Pigouvian pollution tax to the firm's activities to offset the cost of the externality.[7] If left unregulated, the best interest of the decentralized firm is to maximize the private profits ignoring the damage by the pollution so that market price equals to the marginal cost at the optimum.[8] Social optimality is achieved by maximizing the net benefits from extraction that also takes into account the pollution damages. In that case, the marginal benefits should be equal to the social marginal cost that incorporates the pollution damages. If we impose a pollution tax to the firm, which is equal to the marginal damage from pollution, we will get the social optimum. Assume that the regulator can monitor the pollution damages and imposes the Pigouvian pollution tax to the firm, then the firm's objective is given by:

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[5] It is also possible to have quantity restriction of the production activities or residual wastes that causes the pollution damage like in the provision of Kyoto Protocol.

[6] In the benefit side, I use different formulation so that social damage function is incorporated in the objective function.

[7] In fact there are other incentive based instruments: pollution tax, input tax, output tax, tradable permits, abatement subsidy, technology subsidy/tax all would give optimal results under different circumstances.

[8] This is the case in all of the reviewed articles in this paper.

(31) $\text{Max J}_{[z(t),\, q(t)]} = \int_0^T e^{-\hat{a}t} [P(t)\, z(t) - C() - D(W(t))]\, dt$

Subject to:

(32) $\dot{R}(t) = -q(t)$, R(0) = , R(T) e" 0

(33) $\dot{W}(t) = [\acute{a}\, z(t) - \hat{a}\, W(t)]$

Here, equation (31) incorporates the damage function (which is a function of waste generation), as the firm is liable to pay taxes to offset the damage costs. The assumptions for equation (32) are that the resource is depleting at the rate of extraction where there is no uncertainty, no discoveries, and no technological changes over the entire period. For equation (33), we assume that the conversion of gross extraction (*q(t)*) into final product (*z(t)*) generates waste (*W(t)*)[9] at the rate á, (where 1> á > 0), and the waste itself degrades at the rate â (where 1>â > 0). At the terminal time *T*, *z (T)* = 0. The distinctive feature of this alternative formulation of the exhaustible resource models is that the damage function is incorporated into the objective function along with the indirect cost function so that the firm must take into account not only the private extraction cost, but also the social cost that is reflected into the damage function, *D(W (t))*[10]. We also incorporate the transition equation for the waste generation process. In the case of externality, we need some kind of regulatory mechanism to regulate the firms' activities (in the absence of the pollution market), otherwise firm would not take into account the externality effect of residual wastes. The negative externality of the pollutants that originate while processing the extracted exhaustible natural resources for final product and its inclusion in the objective of the firm makes the method of determining the optimal extraction path more complicated than the usual methods in which the externality aspects has been ignored. In this case, the extraction process depends not only on the private activities of the firms, but also in the public policies (tax policy, for example) that change the firm's production behavior. The present value Hamiltonian for the system (31) - (33) is given by:

(33a) $$H = e^{-rt}[P(t)\, z(t) - C(z(t), q(t), V, R(t), t) - D(W(t))] - \ddot{e}_1(t)\, q(t) + \ddot{e}_2(t)[\acute{a}\, z(t) - \hat{a}W(t)]$$

Here $\ddot{e}_1$ is the user cost of the resources, and the $\ddot{e}_2$ is the shadow cost of the stock pollutant. At the optimum, the following necessary conditions should be satisfied.[11]

(34) $$H_Z = e^{-rt}[P - C_Z(\bullet)] + \acute{a}\ddot{e}_2 = 0$$

(35) $$H_q = -e^{-rt} C_q - \ddot{e}_1 = 0$$

(36) $$-H_R = \dot{\lambda}_1 = e^{-rt} C_R$$

(37) $$-H_W = \dot{\lambda}_2 = e^{-rt} D_W + \hat{a}\, \ddot{e}_2$$

[9] Here we are assuming that extraction of q does not generate waste. Waste is generated when the firm produces final output (Z).

[10] Here the damage function reflects the taxes that the polluter firm is liable to pay.

[11] The partial derivative is denoted by the suffix, and the time derivative is denoted by dot (.) over the variable.

(38) $H(T) = e^{-rT}[P(T)\, z(T) - C(T)] - ë_1(T)\, q(T) + ë_2(T)[áz(T) - âW(T)] = 0$

(39 $\overset{\circ}{R}(t) = -q(t)$, $R(0) =$, $R(T)\, e''\, 0$

(40) $\dot{W}(t) = á\, z(t) - â\, W(t)$,

(41) $ë_1(T)\, e''\, 0$ (0 if $R(T) > 0$), and $ë_2(T)\, d''\, 0$ (0 if $W(T) = 0$)

Rearranging equation (34) gives

(42) $ë_2 = -e^{-rt}[P - C_Z(\bullet)] / á$

Where $ë_2 < 0$ as waste (W) is a bad, $P > C_Z(\bullet)$ and $P = C_Z(\bullet) - á\, e^{rt}\, ë_2$. This condition says that market price of the final product should be greater than the MC of producing z due to the imposition of the pollution tax on the production of final product.[12] In this case, the equilibrium extraction rate is less than that under unregulated production assuming that the market demand is downward sloping.

Discussion of the new results

Following propositions summarize the results of this analysis.[13]

Proposition I: *In the extraction of exhaustible resources, if there is no distinction between extracted goods (q) and the final product (Z), then price of the final product is one of the determinants of optimal extraction path and the shadow price.*

Proof: When we disregard the difference between the gross extraction and net final product and in the absence of externality effect of stock pollutant ($\alpha = \beta = 0$), we have the following Hamiltonian:

(43) $$H = e^{-rt}[P(t)q(t) - C(q(t), V, R(t), t)] - \lambda_1(t)q(t)$$

After taking the first order conditions, we get the following extraction path and shadow price of the resource both are dependent on the market price of the final produce:

(44) $$\dot{q} = \frac{rC_q + \dot{P} - rP - C_{qR}\dot{R} - C_{qt} - C_R}{C_{qq}}$$

(45) $$\lambda_1 = e^{-rt}(P - C_q)$$

Proposition II: *In the extraction of exhaustible resources, if there is a distinction between gross extracted goods (q) and the final product (Z) such that q < Z, but the externality effect of waste accumulation is not accounted for [á=â=0], then optimal extraction path and the shadow price both do not depend on price of the final product.*

[12] The second component in CZ (●) - a ert ?2 is the compounded value of the shadow cost of the waste multiplied by the rate of waste production over time from the production of the final good.

[13] Under zero marginal cost of extraction, we will get the Hotelling's r- percent rule, a special case where price increases at the rate of interest: $r = \dot{P}/P$ (Hotelling, 1931).

Proof: If there is distinction between gross extraction and the final product but the pollution aspects of the process is left out, then we have the following Hamiltonian:

(46) $H = e^{-rt}[P(t)Z(t) - C(q(t), Z(t), V, R(t), t)] - \lambda_1(t)q(t)$

The necessary conditions for the optimum give the following shadow price, and the extraction path, none of them are dependent on the final product price (see appendix), the results consistent with Chermak and Patrick (2001, 2002).

(47) $\ddot{e}_1 = -e^{-rt} C_q$

(48) $\dot{q} = \dfrac{rC_q - C_{qz}\dot{Z} - C_{qR}\dot{R} - C_{qt} - C_R}{C_{qq}}$

Proposition III: *In the extraction of exhaustible resources, if there is a distinction between extracted goods (q) and the final product (Z), and also if the production process results into accumulation of stock pollutant (W) that has externality effect (accounted by pollution tax), then the price of the final product is one of the determinant of optimal extraction path and the shadow price of the pollutant.*

Proof: If there is distinction between gross extraction and the final product and externality effect of the pollutions are accounted for, then present value Hamiltonian is the given by (33a). Using maximum principle we find that the production path and the shadow price of the generated wastes both depend on the explicit price of the final product as shown in flowing equations (see appendix for driving (25)):

(49) $\ddot{e}_2 = -e^{-rt}[P - C_Z(\bullet)] / \acute{a}$

(50) $\ddot{e}_1 = -e^{-rt} C_q$

(51) $\dot{q} = \dfrac{C_{zz}(rC_q - C_{qR}\dot{R} - C_{qt} - C_R) + C_{qz}(rP - \dot{P} - \alpha D_W - (r+\beta)C_z + C_{zR}\dot{R} + C_{zt})}{(C_{zz}C_{qq} - C_{qz}C_{zq})}$

The current value formulation of the shadow prices, (49) and (50), are given by:

(49a) $\mu_2 = -\frac{1}{\alpha}(P - C_Z)$ where $\mu_2 = e^{rt}\lambda_2$

(50a) $\mu_1 = -C_q$ where $\mu_1 = e^{rt}\lambda_1$

Using relations (49a) and (50a) we get the following dynamic optimality conditions after taking time derivatives and substituting from (36) and (37).[14]

(52) $\dot{\mu}_1 = r\mu_1 + C_R$

(53) $\dot{\mu}_2 = (r+\beta)\mu_2 + D_w$

Equations (52) and (53) are the basis for testing the dynamic optimality condition of the extraction of exhaustible resources that not only generate final product but also the stock

[14] As $\mu_1 = e^{rt}\lambda_1$, $\dot{\mu}_1 = re^{rt}\lambda_1 + e^{rt}\dot{\lambda}_1$, and $\dot{\lambda}_1 = e^{-rt}C_R$, which results into $\dot{\mu}_1 = r\mu_1 + C_R$ Similarly, as $\mu_2 = e^{rt}\lambda_2$ $\dot{\mu}_2 = re^{rt}\lambda_2 + e^{rt}\dot{\lambda}_2$ and $\dot{\lambda}_2 = e^{-rt}D_W + \beta\lambda_2$, we have $\dot{\mu}_2 = (r+\beta)\mu_2 + D_w$.

pollutant where firm is taxed for the damages due to the pollution from the production activities. Additional complication comes from the fact that we need one extra piece of information to test the theory of exhaustible resources. That means we need the information about the value of marginal damage. This information does not come from the market-based information. We can accomplish this goal by using either the damage value method or the control cost method.

Concluding Remarks

This paper reviews basic exhaustible resources model proposed by Hotelling (1931). Some extensions in the basic models are also considered, and empirical tests about the Hotelling basic principle have been reviewed in which there is no uniform conclusion about the predictive capacity of the Hotelling's model in terms of the firm's behavior or real world data. In general, those tests appeared to be inconclusive whether firms' behaviors are consistent with what the theory predicts. One possible reason of such inconsistent result might be the time series data issues like non-stationarity and co- integration are ignored in all empirical tests.

Another issue in the exhaustible resource models is the omission of the externality aspect of the extraction activities. We propose an extended model of the extraction of exhaustible resources where the gross extraction differs from the final product, and the production process of final good gives rise to a stock pollutant. The stock pollutant is a bad, and it generates an externality to the environments. Assuming that the regulator could determine the value of the marginal damages due to pollution externality, a scheme of pollution tax can be devised based on the marginal damages of pollution. In this case, the firm has to internalize the externality through paying pollution taxes. Under the emissions of stock pollutants and imposition of Pigouvian pollution tax, the extraction path of the exhaustible resources depends on the price path of the final product as well as the change in resource stocks, the indirect marginal costs of q, z and R, and cross partials of the these indirect marginal costs functions; pollution tax rate, and some other parameters. This result is different form other studies reviewed in this paper.

Appendix: Mathematical Derivation

Differentiation (35) with respect to t, we get

$$\dot{\lambda}_1 = re^{-rt}C_q - e^{-rt}[C_{qq}\dot{q} + C_{qz}\dot{Z} + C_{qR}\dot{R} + C_{qt}] \qquad (A1)$$

Substituting for $\dot{\lambda}_1$ in (36), we get

$$\dot{q} = \frac{rC_q - C_{qz}\dot{Z} - C_{qR}\dot{R} - C_{qt} - C_R}{C_{qq}} \qquad (A2)$$

This is the extraction path of resources if we ignore the externality of stock pollutants even if we have .

Now differentiating (34) with respect t, we get an expression for $\overset{\circ}{\lambda}_2$ as

$$\overset{\circ}{\lambda}_2 = \frac{1}{\alpha}[re^{-rt}(P - C_z) - e^{-rt}(\overset{\circ}{P} - C_{zz}\overset{\circ}{Z} - C_{zq}\overset{\circ}{q} - C_{zR}\overset{\circ}{R} - C_{zt}] \qquad \text{(A3)}$$

From (34), we can express λ_2 as

$$\lambda_2 = -\frac{1}{\alpha}e^{rt}(P - C_Z) \qquad \text{(A4)}$$

Substituting for λ_2 and $\overset{\circ}{\lambda}_2$ in (37), we get an expression for the production path of the final product:

$$\overset{\circ}{Z} = \frac{\alpha D_w - (\beta + r)(P - C_z) + \overset{\circ}{P} - C_{zq}\overset{\circ}{q} - C_{zR}\overset{\circ}{R} - C_{zt}}{C_{zz}} \overline{Z} \qquad \text{(A5)}$$

Combining (A2) and (A5) by substituting for in (A2) gives (51).

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