BIBECHANA

A Multidisciplinary Journal of Science, Technology and Mathematics

ISSN 2091-0762 (online)

Journal homepage: http://nepjol.info/index.php/BIBECHANA

Relation between saturated and normal operators

Nagendra Pd. Sah

Dept. of Mathematics, M.M.A.M.Campus, Biratnagar.

.

Article history: Received 6 December, 2013

Abstract

A vector space **X** with algebra of all linear maps **L(X)** from X into itself and the ideal of all finite dimensional linear maps $\sum(\mathbf{X})$with dual (Conjugate) transformation **T*** to **T** from **X'** to itself form a relation in terms of relatively regular and linearly independent which is sufficient for mentioned title.

Keywords: Relatively regular, saturated, normal operator.

1. Introduction

Let X be a vector space, ***L*(X)**be the algebra of all linear map of **X** into itself and $\sum(X)$ be the ideal of all finite dimensional linear maps ***L*(X).** Let **X*** be the algebraic dual space **X** with elements x*, y*,…… and T$\in$ ***L*(X)**, then we define by y*(x) = x*(Tx), x*$\in$ X*,a linear form y*$\in$X*. The map$x^* \rightarrow y^*$is called the (algebraic) dual or conjugate transformation T* to T. It is a linear map of X* into itself and is uniquely characterized by < Tx, x*> = <x, T* x*>, x $\in$ X, x* $\in$ X***[1]**. If X' is a linear subspace of X', therefore a vector space of linear forms, we denote by ***L*(X')** the set of all T$\in$***L*(X),** whose dual transformation T* maps the space X' into itself.

Properties of dual transformation T*

(1) $(T_1+T_2)^* = T_1^* + T_2^*$

(2) $(\alpha T)^* = \alpha T^*$

(3) $(T_1 T_2)^* = T_2^* T_1^*$

Definition (1): If for a continuous operator T, there exists a continuous operator S with T S T = T, then T is called relatively regular.

Definition (2): The algebra $\mathfrak{R}$ of operators on a vector space is called normal, if every finite dimensional operator T from $\mathfrak{R}$ is relatively $\mathfrak{R}$ - regular.

Definition (3): An algebra $\mathfrak{R}$ of operators on vector space X is called saturated, if corresponding to any pair of finite sets $\{x_1,……..x_n\}$, $\{y_1…..y_n\} \subset X$, Where $\{x_1, x_2……..x_n\}$ is linearly independent and there exists T$\in \mathfrak{R}$ with $Tx_\gamma = y_\gamma$, γ= 1,2…….n.

Theorem (1.1): Every operator T$\in$ ***L*(X)** is relatively ***L*(X)**-regular.

Proof: We have X = N(T)$\oplus$U and X = B(T)$\oplus$C. If P is the projection of X onto B(T) along C, Q the projection of X onto N(T) along U and T_o the restriction of T to U, then T_o is a bijective linear map of U onto B(T). Therefore $S = T_o^{-1}P \in \boldsymbol{L}(\mathbf{X})$. For x = n + u, n = Q x ∈N(T), u = (I - Q) x ∈ U, we have

$$
\begin{aligned}
S\,Tx = T_o^{-1}P\,Tx = T_o^{-1}T\,x \\
&= T_o^{-1}T\,(n+u) \\
&= T_o^{-1}Tu \\
&= T_o^{-1}T_o u \\
&= u = (T-Q)x.
\end{aligned}
$$

Therefore, S T = I – Q and T S T = T – TQ = T.
Hence T is relatively regular.

Theorem (1.2): Every saturated operator algebra $\mathfrak{R}$ is normal.

Proof : Let $x_1,\dots\dots,x_n$ be a basis of the image space of $T\in \Sigma(X) \cap \mathfrak{R}$ y_γ, $1\le \gamma \le n$, be so chosen that $T\,y_\gamma = x_\gamma$, $1\le\gamma\le n$, then on account of being saturated, there exist $S\in\mathfrak{R}$ with $S\,x_\gamma = y_\gamma$ and hence also with $T\,S\,x_\gamma = x_\gamma$, $1\le\gamma\le n$. Then for

$$T\,x = \sum_{\gamma=1}^{n} \alpha_\gamma(x) x_\gamma$$

We obtain

$$T\,S\,T\,x = \sum_{\gamma=1}^{n} \alpha_\gamma(x) \mathrm{TS} x_\gamma = \sum_{\gamma=1}^{n} \alpha_\gamma(x) x_\gamma = \mathrm{T}x$$

Hence T is relatively $\mathfrak{R}$ -regular.

Definition (4): If X' be a linear space of linear functionals on X, then we define the set of all finite dimensional maps of the form $\mathrm{T}x = \sum x'_\gamma(x) x_\gamma$, $x'_\gamma \in \mathrm{X}'$ and $x_\gamma \in \mathrm{X}$ by γ (X').
γ (X') is an algebra if E is a topological vector space with dual space E' then γ (E')= **F**(E).

Theorem (1.3): If X is a vector space and X' is a total, then γ (X') is saturated; every super algebra. i.e. γ (X') is normal.

Proof: If $\{x_1,\dots\dots,x_n\}$ is linearly independent subset and $\{y_1,\dots\dots,y_n\}$ arbitrary from X, then by definition. (2) corresponding to the elements $x_1,\dots\dots,x_n$ there exist linear forms $x'_1,\dots\dots,x'_n$ on X' such that $x'_\gamma(x_\mathrm{M}) = \partial_{\gamma\mathrm{M}}$, $1\le\gamma,\ \mathrm{M}\le n$**[2]**. Since x' is total, we define T by

$$T\,x = \sum_{\gamma=1}^{n} x'_\gamma(x) y_\gamma \quad \text{, then}$$

$$
\begin{aligned}
T\in\gamma\,(X') \text{ and } Tx_\mathrm{M} &= \sum_{\gamma=1}^{n} x'_\gamma(x_\mathrm{M}) \\
&= y_\mathrm{M},\ 1\le \mathrm{M} \le n.
\end{aligned}
$$

Hence γ (X') is saturated.

Lemma(1):T*(X') ⊂X'. And ***L*(X')** is an algebra containing I.

Proof: We have γ (X') ⊂ ***L*(X')[4]**
If T∈ γ (X') then

$$\mathrm{T}x = \sum_{\gamma=1}^{n} < x, x'_\gamma > x_\gamma \text{ with } x'_\gamma \in \mathrm{X}' \text{ and } x_\gamma \in \mathrm{X}$$

Then for arbitrary $x'_\gamma \in \mathrm{X}'$, we have

$<x, T^*x'> = <Tx, x'>$

$$= <\sum_{\gamma=1}^{n} < x, x'_{\gamma} > x_{\gamma,} x' >$$

$$= \sum_{\gamma=1}^{n} < x, x'_{\gamma} >< x'_{\gamma,}\ x'_{\gamma} >$$

Therefore $T^*x' = \sum_{\gamma=1}^{n} < x_{\gamma}, x' > x' \in X'$

Theorem (1.4): If X' is a total vector space of linear forms on X, then ***L*(X')** is normal.
The result follows from **Lemma (1)** and theorem **(1.3)** **[3]**.

2. Conclusions

The super algebra of a saturated operator is again saturated **[5]**. Every saturated operators of algebra $\mathfrak{R}$ is normal. If E is a topological vector space with total dual space E', then ***L*(E)** is always saturated. But if ***L*(E)** is not normal for every topological vector space E, then ***L*(E)** is not always saturated **[6]**.

References

1. A.E. Taylor: Introduction to Functional Analysis, New-York John Wiley and Sons Inc. London, 1956.
2. A.P. Robertson and W. Robertson, Topological Vector Spaces Cambridge University Press, 1964.
3. B. Conway John, A Course on Functional Analysis Springer, Verlang, 1985.
4. M.M. Day, Normed Linear Spaces, Berline, Springer, Verlag, 1958.
5. M.M. Day, Normed Linear Spaces, Berlin, Springer, Verlang, 1958.
6. R.G. Cooke, Linear operators, Macmillan, London, 1953.