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Solutions of the Sitnikov's circular restricted three body problems when primaries are oblate spheroid

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Abstract

The solutions of sitnikov's circular restricted three body problem has been tried to obtain by using Lindstedt poincare method if the primaries are oblate spheroid.

Keywords: Lindstedt poincare method, Oblate spheroid, Perturbation, Primaries, Sitnikov's circular restricted three body problem.

1. Introduction

MacMillan [1] has studied an integrable case in restricted problem of three bodies by imposing further restrictions on the restricted three body problem by supposing the two finite bodies of equal masses and an infinitesimal body be moving in their common axis of revolution.

Sitnikov [2] motion in three body problem has two primaries of equal masses and these two moving in (i) circular orbits and (ii) elliptic orbits around their center of mass. The third body (infinitesimal mass) is moving along a line which is perpendicular to the plane of primaries and passing through the center of mass of the primaries.

Further Hagel [3] have studied a higher order perturbation analysis of the Sitnikov problem and investigated the low amplitude bounded oscillatory solutions in full range of primary eccentricities $-0.99 < e < 0.99$. They have found that near integrals in a polynomial form can be obtained for sufficiently small oscillation amplitudes in the entire interval of eccentricities. In addition they derived a relation for the non-linear frequency of the oscillatory solution as function of e and T_0.

Faruque [4] has studied a new analytic expression for the position of the infinitesimal body in the elliptic Sitnikov problem. This solution is valid for small bounded oscillations in cases of moderate primary eccentricities. The final solution to the equation with non-linear force included is obtained through first the use of a courant and Synder transformation followed by Lindstedt-poincare perturbation method and again an application of courant and Synder transformation.

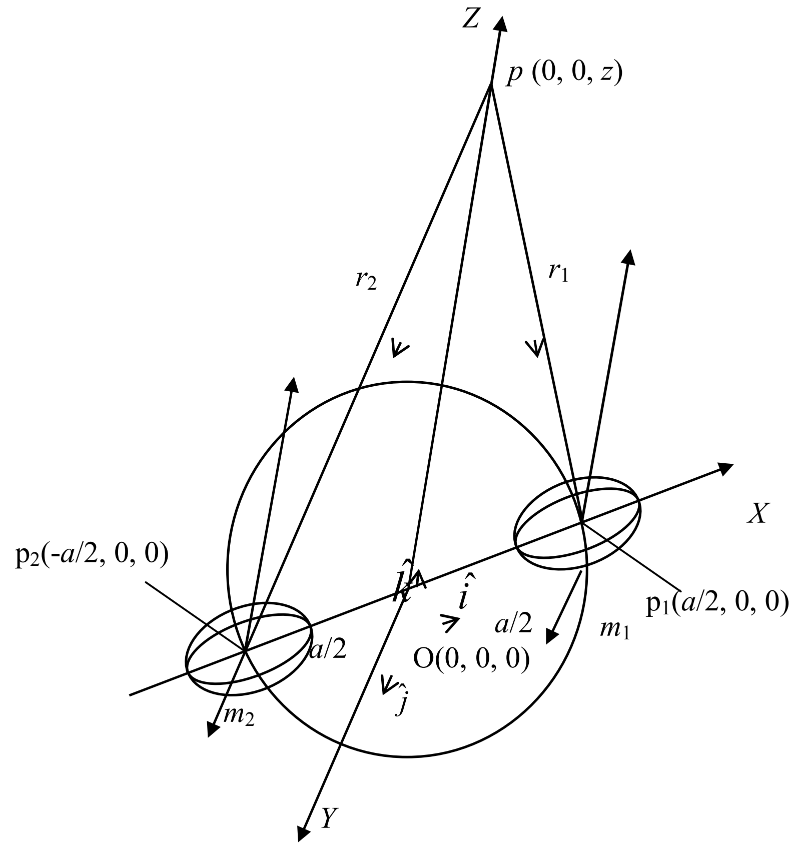
We have studied the Sitnikov's circular restricted problem of the bodies when both the primaries are oblate spheroids and moving in circular orbits around their centre of mass. The Sitnikov's problem is a special case of the restricted three body problem when both the primaries are of equal masses ($m_1 = m_2 = 1/2$) moving in circular orbits or elliptic orbits under Newtonion force of attraction and the third body of mass m_3 (m_3 is much less then the primaries) moves along the line perpendicular to the plane of motion of the primaries and passes through the center of mass of the primaries.

2. Equations of motion

Here the system consists of two oblate primaries with equal masses ($m_1 = m_2 = 1/2$). The third body of mass m_3 is much less than the masses of the primaries. We know that both the primaries in the Sitnikov's problem move on the circumference of the same circle if the primaries are spheroid in shape. But if the primaries are oblate spheroid in shape then due to their oblateness the primaries will not be equidistant from their center of mass. To keep the primaries equidistant from the centre of mass of the primaries, the following conditions are to be imposed :

(i) Principal axes of the oblate bodies should be parallel to the synodic axes.

(ii) The masses of both the primaries should be equal.



Let a_i, b_i, c_i (i = 1, 2) be the semi axes of the primaries. When $a_i = b_i$, as the primaries are oblate, then the moment of inertia of the oblate spheroid m_1 and m_2 about the principal axes of body are

$$\left.\begin{aligned} A_1 = A_2 &= \frac{b^2+c^2}{5} = \frac{a^2+c^2}{5} \\ B_1 = B_2 &= \frac{a^2+c^2}{5} \\ C_1 = C_2 &= \frac{a^2+b^2}{5} = \frac{2a^2}{5} \end{aligned}\right\} \tag{1}$$

Thus the potential between two bodies m_1 and m_2 is given by

$$\begin{aligned} -v = \frac{Gm_1m_2}{r} &+ \frac{Gm_1}{r^3}\left[\frac{A_1+B_1+C_1}{2} - \frac{3}{2}\left(A_1l_1^2 + B_1m_1^2 + C_1n_1^2\right)\right] \\ &+ \frac{Gm_2}{r^3}\left[\frac{A_2+B_2+C_2}{2} - \frac{3}{2}\left(A_2l_2^2 + B_2m_2^2 + C_2n_2^2\right)\right] \end{aligned} \tag{2}$$

$$\Rightarrow -v = \frac{Gm_1m_2}{r} + \frac{GA(m_1+m_2)}{r^3} \tag{3}$$

where $A = \dfrac{a^2-c^2}{10}$ = oblatenless of the primaries

If $op_1 = r_1$ and $op_2 = r_2$ and $\hat{u}_1$ and $\hat{u}_2$ are the unit vectors along r_1 and r_2, then the equation of motion of m_1 can be written as

$$\Rightarrow \left(-n^2 + \frac{1}{2r^3} + \frac{4A}{r^5}\right) r_1\hat{u}_1 - \left(\frac{1}{2r^3} + \frac{4A}{r^5}\right) r_2\hat{u}_2 = 0 \tag{4}$$

Similarly from equation of motion of m_2, we get

$$-\left(\frac{1}{2r^3} + \frac{4A}{r^5}\right) r_1\hat{u}_1 + \left(-n^2 + \frac{1}{2r^3} + \frac{4A}{r^5}\right) r_2\hat{u}_2 = 0 \tag{5}$$

$$\Rightarrow r_1\hat{u}_1 + r_2\hat{u}_2 = 0 \tag{6}$$

As Mccuskey [5] the equations (4), (5) and (6) have the non-trivial solutions if the 2×2 determinant obtained from any two of the above three equations is zero. i.e.

$$\Rightarrow n^2 = \frac{1}{r^3} + \frac{8A}{r^5} \tag{7}$$

Now following Suraj et.al.[6], we can find the equation of motion of the third body as

$$\frac{d^2z}{dt^2} = \frac{-z}{(z^2+r^2)^{3/2}} - \frac{9Az}{(z^2+r^2)^{5/2}} + \frac{15Az^2}{(z^2+r^2)^{7/2}}$$

$$\Rightarrow \left(1 + \frac{8A}{3}\right)^2 = 1 + \frac{16A}{3} = b \text{, say}$$

$$\Rightarrow \frac{d^2z}{dt^2} + \frac{z}{(z^2+b)^{3/2}} + \frac{9Az}{(z^2+b)^{5/2}} - \frac{15Az^2}{(z^2+b)^{7/2}} = 0 \tag{8}$$

3. Solutions by Lindsted-Poincare Method

The equation of motion of third body is

$$\frac{d^2z}{dt^2} + \frac{z}{\left(z^2+b\right)^{3/2}} + \frac{9Az}{\left(z^2+b\right)^{5/2}} - \frac{15Az^2}{\left(z^2+b\right)^{7/2}} = 0$$

$$\Rightarrow \frac{d^2z}{dt^2} + \left(\frac{1}{b^{3/2}} + \frac{9A}{b^{5/2}}\right)z - \frac{15A}{b^{7/2}}z^2 - \left(\frac{3}{2b^{5/2}} + \frac{45A}{2b^{7/2}}\right)z^3$$
$$+ \frac{105Az^4}{2b^{9/2}} + \left(\frac{15}{8b^{7/2}} + \frac{315}{8b^{9/2}}\right)z^5 - \ldots = 0$$

$$\begin{aligned}&\Rightarrow \frac{d^2z}{dt^2} + \left(1+A\right)z - 15Az^2 - \frac{3}{2}\left(1+\frac{5A}{3}\right)z^3 \\ &+ \frac{105A}{2}z^4 + \frac{15}{4}\left(11 - \frac{784A}{3}\right)z^5 + \ldots = 0\end{aligned} \tag{9}$$

Since, there is no effect of independent perturbating terms i.e. $15Az^2$, $\frac{105Az^4}{2}$ and hence they may be omitted from the equation (9)

We get, $\frac{d^2z}{dt^2} + \left(1+A\right)z - \frac{3}{2}\left(1+\frac{5A}{3}\right)z^3 + \ldots = 0$

$$\Rightarrow \frac{d^2z}{dt^2} + \eta_0^2 z - \varepsilon z^3 = 0 \tag{10}$$

Where, $\eta_0^2 = 1 + A$ and $\varepsilon = \frac{3}{2}(1+\frac{5}{3}A)$

Let us write,

$$z(t) = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + \varepsilon^3 z_3(t) + \ldots$$

& $\tau = \eta t$, where η is some parameter

$$\frac{d\tau}{dt} = \eta$$

Let us write $\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \ldots$

From (10)

$$\eta^2 \frac{d^2 z}{d\tau^2} + \eta_0^2 z - \varepsilon z^3 = 0$$

$$\Rightarrow \left(\eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + ...\right)^2 \frac{d^2}{d\tau^2}\left(z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + ...\right)$$

$$+\eta_0^2\left(z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + ...\right) - \varepsilon\left(z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_3 + ...\right)^3 = 0$$

$$\Rightarrow \eta_0^2\left(\frac{d^2 z_0}{d\tau^2} + z_0\right) + \varepsilon\left(\eta_0^2 \frac{d^2 z_1}{d\tau^2} + \eta_0^2 z_1 + 2\eta_1\eta_0 \frac{d^2 z_0}{d\tau^2} - \alpha_0\right)$$

$$+\varepsilon^2\left(\eta_0^2 \frac{d^2 z_2}{d\tau^2} + \eta_0^2 z_2 - \alpha_1 + 2\eta_1\eta_0 \frac{d^2 z_1}{d\tau^2} + \left(\eta_1^2 + 2\eta_2\eta_0\right)\frac{d^2 z_0}{d\tau^2}\right)$$

$$+\varepsilon^3\left[\eta_0^2 \frac{d^2 z_3}{d\tau^2} + \eta_0^2 z_3 - \alpha_2 + 2\eta_1\eta_0 \frac{d^2 z_2}{d\tau^2} + \left(\eta_1^2 + 2\eta_2\eta_0\right)\frac{d^2 z_1}{d\tau^2}\right.$$

$$\left. +2\left(\eta_1\eta_2 + \eta_3\eta_0\right)\frac{d^2 z_0}{d\tau^2}\right] + ... = 0 \qquad (11)$$

Hence (11) is identity in ε so the coefficients of $\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, ...$ must be zero.

$$\therefore \frac{d^2 z_0}{d\tau^2} + z_0 = 0 \qquad (12)$$

$$\eta_0^2\left(\frac{d^2 z_1}{d\tau^2} + z_1\right) + 2\eta_1\eta_0 \frac{d^2 z_0}{d\tau^2} - z_0^3 = 0 \qquad (13)$$

$$\eta_0^2\left(\frac{d^2 z_2}{d\tau^2} + z_2\right) + \left(\eta_1^2 + 2\eta_2\eta_0\right)\frac{d^2 z_0}{d\tau^2} + 2\eta_1\eta_0 \frac{d^2 z_1}{d\tau^2} - 3z_0^2 z_1 = 0 \qquad (14)$$

$$\eta_0^2\left(\frac{d^2 z_3}{d\tau^2} + z_3\right) + 2\left(\eta_1\eta_2 + \eta_3\eta_0\right)\frac{d^2 z_0}{d\tau^2} + \left(\eta_1^2 + 2\eta_2\eta_0\right)\frac{d^2 z_1}{d\tau^2}$$

$$+2\eta_1\eta_0 \frac{d^2 z_2}{d\tau^2} - 3\left(z_0^2 z_2 + z_0 z_1^2\right) = 0 \qquad (15)$$

The general solution of the equation (12) is

$z_0 = c_1 \cos \tau + c_2 \sin \tau$

$$z_0 = c \cos \tau \qquad (16)$$

From (16),

$$\eta_0^2\left(\frac{d^2 z_1}{d\tau^2} + z_1\right) + 2\eta_1\eta_0 \frac{d^2 z_0}{d\tau^2} - z_0^3 = 0$$

To avoid the secular terms, equating the coefficient of cos τ to zero,

we get

$$\eta_1 = \frac{-3c^2}{8\sqrt{(1+A)}} \tag{17}$$

$$\frac{d^2 z_1}{d\tau^2} + z_1 = \frac{c^3}{4\eta_0^2}\cos 3\tau \tag{18}$$

The general solution of the equation (18) is

$$z_1 = c_3\cos\tau + c_4\sin\tau - \frac{c^3}{32\eta_0^2}\cos 3\tau$$

$$\therefore z = z_0 + \varepsilon z_1$$

$$z = c\cos\tau + \varepsilon c_3\cos\tau + \varepsilon c_4\sin\tau - \frac{\varepsilon c^3}{32\eta_0^2}\cos 3\tau$$

$$\dot{z} = -c\sin\tau - \varepsilon c_3\sin\tau + \varepsilon c_4\cos\tau + \frac{3\varepsilon c^3}{32\eta_0^2}\sin 3\tau$$

$$\therefore z = c\cos\tau + \varepsilon\frac{c^3}{32\eta_0^2}\cos\tau - \frac{\varepsilon c^3}{32\eta_0^2}\cos 3\tau \tag{19}$$

or, $$z_1 = \frac{c^3}{32\eta_0^2}(\cos\tau - \cos 3\tau) \tag{20}$$

$$z_1 = \frac{c^3}{32(1+A)}(\cos\tau - \cos 3\tau) \tag{21}$$

$$\frac{d^2 z_1}{d\tau^2} = \frac{c^3}{32(1+A)}(9\cos 3\tau - \cos\tau)$$

Now from (14)

$$\eta_0^2\left(\frac{d^2 z_2}{d\tau^2} + z_2\right) + (\eta_1^2 + 2\eta_2\eta_0)\frac{d^2 z_0}{d\tau^2} + 2\eta_1\eta_0\frac{d^2 z_1}{d\tau^2} - 3z_0^2 z_1 = 0$$

$$\eta_2 = \frac{-21c^4}{256(1+A)^{3/2}}$$

and $z_2 = c_5 \cos\tau + c_6 \sin\tau - \frac{3c^5}{128\eta_0^4}\cos 3\tau + \frac{c^5}{1024\eta_0^4}\cos 5\tau$

since, $z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2$

$$z = c\cos\tau + \frac{\varepsilon c^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right)$$

$$+\varepsilon^2\left(c_5\cos\tau + c_6\sin\tau - \frac{3c^5}{128\eta_0^4}\cos 3\tau + \frac{c^5}{1024}\frac{\cos 5\tau}{\eta_0^4}\right)$$

$$\therefore z = c\cos\tau + \frac{\varepsilon c^3}{32(1+A)}\left(\cos\tau - \cos 3\tau\right)$$

,

$$+\varepsilon^2\frac{23c^5}{1024(1+A)^2}\cos\tau - \frac{3c^5\varepsilon^2}{128\eta_0^4}\cos 3\tau + \frac{c^5\varepsilon^2\cos 5\tau}{1024\eta_0^4}$$

$$z = c\cos\tau + \frac{\varepsilon c^3}{32\eta_0^2}\left(\cos\tau - 3\cos 3\tau\right) + \frac{\varepsilon^2 c^5}{1024\eta_0^4}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right)$$

Now following, Raju Ram Thapa et.al.[6],

$$z = c\cos\tau + \frac{\varepsilon c^3}{32\eta_0^2}\left(\cos\tau - \cos 3\tau\right) + \frac{\varepsilon^2 c^5}{1024\eta_0^4}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right)$$
$$+\varepsilon^3\left[\frac{547}{32768}\frac{c^7}{\eta_0^6}\cos\tau + \frac{c^7}{2048\eta_0^6}\left(\frac{-297}{8}\cos 3\tau + 3\cos 5\tau - \frac{1}{16}\cos 7\tau\right)\right]$$

But, $\eta_0^2 = 1 + A$ and $\varepsilon = \frac{3}{2}\left(1 + \frac{5}{3}A\right)$

$$z = c\cos\tau + \frac{\frac{3}{2}\left(1+\frac{5}{3}A\right)c^3}{32(1+A)}\left(\cos\tau - \cos 3\tau\right)$$
$$+\frac{\frac{9}{4}\left(1+\frac{5}{3}A\right)^2 c^5}{1024(1+A)^2}\left(23\cos\tau - 24\cos 3\tau + \cos 5\tau\right)$$
$$+\frac{27}{8}\left(1+\frac{5}{3}A\right)^3\left[\frac{547}{32768}\frac{c^7}{(1+A)^3}\cos\tau\right.$$
$$\left.+\frac{c^7}{2048\left(1+\frac{5}{3}A\right)^3}\left(\frac{-297}{8}\cos 3\tau + 3\cos 5\tau - \frac{1}{16}\cos 7\tau\right)\right]$$

$$
\begin{aligned}
z &= c\cos\eta t + \frac{3}{64}\left(\cos\eta t - \cos 3\eta t\right)c^3 + \frac{9c^5}{4096}\left(23\cos\eta t - 24\cos 3\eta t + \cos 5\eta t\right) \\
&+ \frac{27c^7}{8}\left[\frac{547}{32768}\cos\eta t - \frac{297}{16384}\cos 3\eta t + \frac{3}{2048}\cos 5\eta t - \frac{1}{32768}\cos 7\eta t\right] \\
&+ A\left[\frac{1}{32}\left(\cos\eta t - \cos 3\eta t\right)c^3 + \frac{3}{1024}\left(23\cos\eta t - 24\cos 3\eta t + \cos 5\eta t\right)c^5\right. \\
&\left.+ \frac{27}{4}\left(\frac{547}{32768}\cos\eta t - \frac{297}{16384}\cos 3\eta t + \frac{3}{2048}\cos 5\eta t - \frac{1}{32768}\cos 7\eta t\right)c^7 + \ldots\right]
\end{aligned}
$$

4. Conclusion

The solution Z of Sitnikov's circular restricted three body problem depends on constant 'C', independent variable τ & oblateness parameter A. The third body moves around the equilibrium point.

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Reference

1. W.D. Macmillan, Astronomical Journal, 27(1913) 11.
2. K.A. Stinikov, Dokl.Akad.Nauk, USSR, 133(1960) 303-306.
3. J. A. Hagel, Celestial mechanics and Dynamical Astronomy, 56 (1993) 81.
4. S.B. Faraque, Celes. Mech & Dyn. Astron., 87(2003) 353-369.
5. S.W. Mccuskey, Introduction to Cel. Mechanics, Addison-Wesley publishing company, inc., 1963
6. M.S. Suraj, and M.R. Hassan, Solution of Sitnikov restricted four body problem when all the primaries are oblate bodies: Circular case, proceeding of PAS, 50 (1)(2013) 61-79.
7. R.R. Thapa and M.R. Hassan, Solutions of Sitnikov's restricted three body problem when the primaries are sources of radiation. Modern trends in science and Technology Devendra Adhikari, Shiva Kumar Rai and Kul Prasad Limbu (editors). Nepal Physical Society (Eastern Chapter) and Research Council of Science and Technology (Publishers), (2013)97-111.