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On 2 - normed space valued paranormed null sequence space $(c_0 (S, \bar{\alpha}, \bar{u} , ||. , .||),T)$

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Abstract

In this paper we introduce and study a new class $c_0 (S, \bar{\alpha}, \bar{u} , ||. , .||)$ of sequences with values in 2- normed space as a generalization of basic null sequence space c_0. We investigate some conditions pertaining to the containment relations of the class $c_0 (S, \bar{\alpha}, \bar{u} , ||. , .||)$ in terms of different $\bar{\alpha}$ *and* $\bar{u}$ and explore the linear topological structures of the space $c_0 (S, \bar{\alpha}, \bar{u} , ||. , .||)$ by endowing it with a suitable natural paranorm.

Keywords: 2- normed space, 2–Banach space, Paranormed space, Sequence Space.

1. Introduction and Preliminaries

Before proceeding with the main results, we recall some of the basic notations and definitions that are used in this paper. The notion of 2–normed space was initially introduced by S. GÄahler [1] as an interesting linear generalization of a normed linear space, which was further studied by J.A. White and Y.J. Cho [2], K.Iseki [3], R. Freese *et al.* [4,5], W.Raymond *et al.* [6] and many others. Recently a lot of activities have been started by many researchers to study this concept in different directions, for instances E. Savas [7], H. Gunawan and H. Mashadi [8] , J.K. Srivastava and N.P. Pahari [9] , M. Açikgöz [10] and others.
Let S be a vector space of dimension $d > 1$ over K, the field of real or complex numbers. A 2 - norm on S is a real valued function $||. , .||$ on $S \times S$ satisfying the following conditions:

2- $N1$: $|| \xi, \eta || \geq 0$ and $|| \xi, \eta || = 0$ if and only if ξ and η are linearly dependent;

2-$N2$: $|| \xi, \eta || = || \eta, \xi ||$, for all $\xi, \eta \in S$;

2-$N3$: $|| \gamma\xi, \eta || = |\gamma| \, ||\xi, \eta ||$, where $\gamma \in K$ and $\xi, \eta \in S$;and

2-N4 : $|| \xi_1 + \xi_2, \eta || \leq || \xi_1, \eta || + || \xi_2, \eta ||$,for all ξ_1 , ξ_2 and $\eta \in S$.

The pair $(X, ||. , .||)$ is called a 2–normed space. Thus the notion of 2–normed space is just a two-dimensional analogue of a normed space . Recall that $(X, ||. , .||)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x_0 in X.

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Geometrically, a 2-norm function generalizes the concept of area function of parallelogram due to the fact that, in the standard case, it represents the area of the usual parallelogram spanned by the two associated vectors.For example, consider $S = \boldsymbol{R}^2$, being equipped with

$$\| \xi, \eta \| = | \xi_1\eta_2 - \xi_2\eta_1|, \text{ where } \xi = (\xi_1, \xi_2) \text{ and } \eta = (\eta_1 , \eta_2).$$

Then $(S, \|. , .\|)$ forms a 2–normed space and $\| \xi, \eta \|$ represents the area of the parallelogram spanned by the two associated vectors ξ and η.

Again, let $S = \boldsymbol{R}^3$ and define the function $\|. , .\|$ on $S \times S$ by

$$\| \xi, \eta \| = \left| \text{Det} \begin{pmatrix} i & j & k \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix} \right|$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$.

Obviously $(S, \|. , .\|)$ forms a 2–normed space.

Analogously, if $(S , <. , .>)$ be a finite dimensional inner product space and $\|. , .\|$ defined on $S \times S$ by

$$\| \xi, \eta \| = \sqrt{\begin{vmatrix} <\xi,\xi> & <\xi, \eta> \\ <\eta, \xi> & <\eta, \eta> \end{vmatrix}},$$

then we can see that $(S, \|. , .\|)$ satisfies all the conditions of 2–normed space.

The notion of convergence has introduced by White and Cho [2]. A sequence (ξ_n) in a linear 2–normed space S is *convergent* if there is an $\xi \in S$ such that $\lim_{n\to\infty} \| \xi_n - \xi, \eta \| = 0$, for each $\eta \in S$ .It is said to be a *Cauchy* if there are η and ν in S such that η and ν are linearly independent and

$$\lim_{m,n\to\infty} \|\xi_m - \xi_n, \eta\| = 0$$

and

$$\lim_{m,n\to\infty} \|\xi_m - \xi_n, \nu\| = 0.$$

A linear 2–normed space $(S, \|.,.\|)$ is called *2–Banach* space if every Cauchy sequence in S is convergent to some $\xi \in S$.

The notion of paranormed space is closely related to linear metric space , see A. Wilansky [11]. A paranormed space (S, T) is a linear space S with zero element θ together with a function $T : S \to \boldsymbol{R}_+$ (called a paranorm on S) which satisfies the following axioms:

*PN*1: $T(\theta) = 0$;

*PN*2: $T(\xi) = T(-\xi)$, for all $\xi \in S$;

*PN*3: $T(\xi + \eta) \leq T(\xi) + T(\eta)$, for all $\xi , \eta \in S$; and

*PN*4: Scalar multiplication is continuous i.e., if (γ_n) is a sequence of scalars with $\gamma_n \to \gamma$ *as* $n \to \infty$ and (ξ_n) is a sequence of vectors with $T(\xi_n - \xi) \to 0$ as $n \to \infty$ then

$$T(\gamma_n \xi_n - \gamma\xi) \to 0 \text{ as } n \to \infty.$$

Note that the continuity of scalar multiplication is equivalent to

(i) if $T(\xi_n) \to 0$ and $\gamma_n \to \gamma$ as $n \to \infty$, then $T(\gamma_n \xi_n) \to 0$ as $n \to \infty$

and

(ii) if $\gamma_n \to 0$ as $n \to \infty$ and ξ be any element in S, then $T(\gamma_n \xi) \to 0$, see Wilansky [11].

A paranorm is called total if $T(\xi) = 0$ implies $\xi = \theta$, see A. Wilansky [11].

The studies of paranorm on sequence spaces were initiated by Maddox [12] and many others. Srivastava and *et al.* [13-15], Pahari [16], Parashar and Choudhary [17], Bhardwaj and Bala [18] and many others further studied various types of paranormed sequence spaces and function spaces.

A sequence space S is said to be *normal* if $\bar{\xi} = (\xi_k) \in S$ and $\bar{\alpha} = (\alpha_k)$ a sequence of scalars with $|\alpha_k| \le 1$, for all $k \ge 1$, then

$$\bar{\alpha}\,\bar{\xi} = (\alpha_k\, \xi_k) \in S .$$

2. The Class $c_0(X, \bar{\lambda}, \bar{u}, \|., .\|)$ of 2-Normed Space Valued Vector Sequences

Let $\bar{u} = (u_k)$ and $\bar{v} = (v_k)$ be any sequences of strictly positive real numbers and $\bar{\lambda} = (\lambda_k)$ and $\bar{\mu} = (\mu_k)$ be the sequences of non zero complex numbers. Let $(X, \|., .\|)$ be the 2-Normed space over the field $\boldsymbol{C}$ of complex numbers and θ denotes the zero element of X. Let $\omega(X)$ denotes the linear space of all sequences $\bar{x} = (\xi_k)$ and $\bar{y} = (\eta_k)$ with $\xi_k, \eta_k \in X$, $k \ge 1$ with usual coordinate wise operations i.e.,

$$\bar{x} + \bar{y} = (\xi_k + \eta_k) \text{ and } \alpha\bar{x} = (\alpha\xi_k) \text{ ,for each } \bar{x}, \bar{y} \in \omega(X) \text{ and } \alpha \in \boldsymbol{C}.$$

We shall denote $\omega(\boldsymbol{C})$ by ω. Further, $\bar{\lambda} = (\lambda_k) \in \omega$ and $\bar{x} \in \omega(X)$ we shall write $\bar{\lambda}\bar{x} = (\lambda_k \xi_k)$.

We now introduce the following classes of 2-normed space X–valued sequences

$$c_0(X, \bar{\lambda}, \bar{u}, \|., .\|) = \{\bar{x} = (x_k) \in \omega(X), x_k \in X, k \ge 1 \text{ satisfying } \|\lambda_k x_k, y\|^{u_k} \to 0 \text{ as } k \to \infty, \text{ for each } y \in X\}.$$

In fact, this class is a generalization of the familiar sequence spaces, studied in Srivastava *et al.* (1996) [12], Srivastava (1996), Pahari (2011) [16], Pahari and et al (2011) [11] , using 2-norm . Further, when $\lambda_k = 1$ for all k , then $c_0(X, \bar{\lambda}, \bar{u}, \|., .\|)$ will be denoted by $c_0(X, \bar{u}, \|., .\|)$ and when $u_k = 1$ for all k, then $c_0(X, \bar{\lambda}, \bar{u}, \|., .\|)$ will be denoted by $c_0(X, \bar{\lambda}, \|., .\|)$. If $u_k = \lambda_k = 1$ for all k , then the class $c_0(X, \bar{\lambda}, \bar{u}, \|., .\|)$ will be denoted by $c_0(X, \|., .\|)$. Further, when $X = \boldsymbol{C}$ we simply write $c_0(X, \bar{\lambda}, \bar{u}, \|., .\|)$ as $c_0(\bar{\lambda}, \bar{u}, \|., .\|)$.

3. Containment Relations

In this section, we investigate some inclusion relations of the class $c_0(S, \|., .\|, \bar{\alpha}, \bar{u})$ arising in terms of different $\bar{u}$ and $\bar{\alpha}$. Throughout, we denote

$$\sup u_k = L \text{ for each } k \text{ and for scalar } \alpha,\ A[\alpha] = \max(1, |\alpha|).$$

But when the sequences u_k and v_k occur, then to distinguish L we use the notations $L(u)$ and $L(v)$ respectively.

Theorem 3.1: For any $\bar{u} = (u_k)$, $c_0(S, \bar{\alpha}, \bar{u}, \|., .\|) \subset c_0(S, \bar{\beta}, \bar{u}, \|., .\|)$ if and only if $\liminf_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} > 0$.

Proof:

For the necessity, assume that

$$c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\beta}, \bar{u}, \|.,.\|)$$

but $\liminf_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} = 0$. Then we can find a sequence $(k(n))$ of integers such that

$$1 \le k(n) < k(n+1), n \ge 1$$

for which

$$n^2 |\alpha_{k(n)}|^{u_{k(n)}} < |\beta_{k(n)}|^{u_{k(n)}}, \text{ for all } n \ge 1. \quad \text{...(3.1)}$$

Now, corresponding to $\rho \in S$ and $\rho \neq \theta$, we define the sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1}\, n^{-2/u_{k(n)}}\rho, \text{ if } k = k(n),\ n \ge 1 \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \quad \text{...(3.2)}$$

Then for $k = k(n)$, $n \ge 1$, we have

$$\|\alpha_k \xi_k, \eta\|^{u_k} = \|n^{-2/u_{k(n)}}\rho, \eta\|^{u_{k(n)}}$$
$$\le \frac{1}{n^2}\|\rho, \eta\|^{u_{k(n)}}$$
$$\le \frac{1}{n^2} A[\|\rho, \eta\|^{L(u)}] \to 0 \text{ as } n \to \infty,$$

where $\|\rho, \eta\|^{u_{k(n)}} \le A[\|\rho, \eta\|^{L(u)}]$ for each $n \ge 1$ is used

and

$$\|\alpha_k \xi_k, \eta\|^{u_k} = 0, \text{ for } k \ne k(n), n \ge 1.$$

This shows that $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$.

On the other hand, let us choose $\eta \in S$ such that $\|w, \eta\| = 1$. Then for $k = k(n)$, $n \ge 1$, and in view of (3.1) and (3.2), we have

$$\|\beta_k \xi_k, \eta\|^{u_k} = \|\beta_{k(n)}\xi_{k(n)}, \eta\|^{u_{k(n)}}$$
$$= \frac{1}{n^2}\|w, \eta\|^{u_{k(n)}} \left|\frac{\alpha_{k(n)}}{\beta_{k(n)}}\right|^{u_{k(n)}} \ge 1.$$

This shows that $\bar{\xi} \notin c_0(S, \bar{\beta}, \bar{u}, \|.,.\|)$, a contradiction.

For the sufficiency, assume that $\liminf_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} > 0$. Then there exists $m > 0$ such that

$$m|\beta_k|^{u_k} < |\alpha_k|^{u_k}$$

for all sufficiently large values of k.

Let $\bar{\xi} = (\xi_k) \in c_0(X, \bar{\alpha}, \bar{u}, \|.,.\|)$. Then for each $\eta \in S$,

$$\|\alpha_k \xi_k, \eta\|^{u_k} \to 0 \text{ as } k \to \infty.$$

Now for each $\eta \in S$,

$$\|\beta_k \xi_k, \eta\|^{u_k} \le \frac{|\alpha_k|^{u_k}}{m}\|\xi_k, \eta\|^{u_k}$$
$$\le \frac{1}{m}\|\alpha_k \xi_k, \eta\|^{u_k} \to 0 \text{ as } k \to \infty.$$

This clearly implies that $\bar{\xi} \in c_0(S, \bar{\beta}, \bar{u}, \|.,.\|)$ and hence

$$c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\beta}, \bar{u}, \|.,.\|).$$

This completes the proof.

Theorem 3.2: For any $\bar{u} = (u_k)$, $c_0(S, \bar{\beta}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ if and only if

$$\limsup_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} < \infty.$$

Proof:

For the necessity**,** suppose that

$$c_0\,(S,\ \bar{\beta}, \bar{u}, \|.\,,.\|) \subset c_0\,(S,\ \bar{\alpha}, \bar{u}, \|.\,,.\|)$$

but $\lim\ \sup_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} = \infty$. Then there exists a sequence $(k(n))$ of positive integers with

$$1 \le k(n) < k(n+1)\ ,\ n \ge 1$$

satisfying

$$|\alpha_{k(n)}|^{u_{k(n)}} > n^2\,|\beta_{k(n)}|^{u_{k(n)}}\ ,\text{for each } n \ge 1. \qquad ...(3.3)$$

Corresponding to $\rho \in S$ and $\rho \neq \theta$, we define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \beta_{k(n)}^{-1}\, n^{-2/u_{k(n)}}\,\rho, & \text{if } k = k(n),\ n \ge 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \qquad ...(3.4)$$

Then for $k = k(n)$, $n \ge 1$, we have

$$\begin{aligned} \|\beta_k\,\xi_k, \eta\|^{u_k} &= \|\beta_{k(n)}\ \xi_{k(n)}\ ,\eta\|^{u\,k(n)} \\ &= \|n^{-2/u_{k(n)}}\rho, \eta\|^{u_{k(n)}} \\ &\le \frac{1}{n^2}\,\|\rho, \eta\|^{u_{k(n)}} \\ &\le \frac{1}{n^2}\,A\,[\,\|\rho, \eta\|^{L(u)}\,] \to 0 \text{ as } n \to \infty \end{aligned}$$

and

$$\|\beta_k\,\xi_k, \eta\|^{u_k} = 0, \text{ for } k \neq k(n)\ ,\ n \ge 1.$$

This shows that $\bar{\xi} \in c_0\,(S,\ \bar{\beta}, \bar{u}, \|.\,,.\|)$.

On the other hand, let us choose $\eta \in S$ such that $\|\rho, \eta\| = 1$. Then for $k = k(n)$, $n \ge 1$, in view of (3.3) and (3.4) ,we have

$$\|\alpha_k\,\xi_k\ ,\eta\|^{u_k} = \left|\frac{\alpha_{k(n)}}{\beta_{k\,(n)}}\right|^{u_{k(n)}} \cdot \frac{1}{n^2}\,\|\rho, \eta\|^{u_{k(n)}} \ \ge\ 1$$

and so $\bar{\xi} \notin c_0\,(S,\ \bar{\alpha}, \bar{u}, \|.\,,.\|)$,a contradiction.

For the sufficiency, assume that $\lim\ \sup_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} < \infty$. Then we can find a positive constant d such that

$$d\,|\beta_k|^{u_k} > |\alpha_k|^{u_k}$$

for all sufficiently large values of k. Let $\bar{\xi} = (\xi_k) \in c_0\,(S,\ \bar{\beta}, \bar{u}, \|.\,,.\|)$. Then we have

$$\|\beta_k\,\xi_k, \eta\|^{u_k} \to 0 \text{ as } k \to \infty\ ,\ \text{ for each } \eta \in S.$$

Now we have

$$\begin{aligned} \|\alpha_k\,\xi_k, \eta\|^{u_k} &\le d|\beta_k|^{u_k}\ \|\xi_k, \eta\|^{u_k} \\ &\le \|\beta_k\ \xi_k, \eta\|^{u_k} \to 0 \text{ as } k \to \infty\ , \end{aligned}$$

for each $\eta \in S$. This clearly implies that $\bar{\xi} \in c_0\,(S,\ \bar{\alpha}, \bar{u}, \|.\,,.\|)$ and hence

$$c_0\,(S,\ \bar{\beta}, \bar{u}, \|.\,,.\|) \ \subset\ c_0\,(S,\ \bar{\alpha}, \bar{u}, \|.\,,.\|).$$

The proof is now complete.

On combining the Theorems 3.1 and 3.2, we get:

Theorem 3.3: For any $\bar{u} = (u_k)$, $c_0\,(S,\ \bar{\alpha}, \bar{u}, \|.\,,.\|) = c_0\,(S,\ \bar{\beta}, \bar{u}, \|.\,,.\|)$ if and only if

$$0 < \lim\ \inf_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} \le \lim\ \sup_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} < \infty.$$

Corollary 3.4: For any $\bar{u} = (u_k)$,

(i) $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{u}, \|.,.\|)$ **if and only if** $\liminf_k |\alpha_k|^{u_k} > 0$ **;**

(ii) $c_0(S, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$
if and only if $\limsup_k |\alpha_k|^{u_k} < \infty$**;and**

(iii) $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) = c_0(S, \bar{u}, \|.,.\|)$
if and only if
$0 < \liminf_k |\alpha_k|^{u_k} \le \limsup_k |\alpha_k|^{u_k} < \infty$.

Proof:

The proof follows by putting $\beta_k = 1$ for all k in Theorems 3.1, 3.2 and 3.3.

Theorem 3.5: For any $\bar{\alpha} = (\alpha_k)$, $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\alpha}, \bar{v}, \|.,.\|)$ **if and only if**

$$\liminf_k \frac{v_k}{u_k} > 0.$$

Proof:

For the necessity, suppose that the inclusion holds but $\liminf_k \frac{v_k}{u_k} = 0$. Then there exists a sequence $(k(n))$ of positive integers such that

$$1 \le k(n) < k(n+1),\ n \ge 1$$

for which

$$n\, v_{k(n)} < u_{k(n)}\ ,\ \text{for each } n \ge 1. \qquad \ldots(3.5)$$

Let $\rho \in S$ and $\rho \ne \theta$. We define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1}\, n^{-1/u_{k(n)}}\rho, \text{ for } k = k(n),\ n \ge 1 \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \qquad \ldots(3.6)$$

Then for each $\eta \in S$, $k = k(n)$, $n \ge 1$, we have

$$\|\alpha_k \xi_k, \eta\|^{u_k} = \|n^{-1/u_{k(n)}}\rho, \eta\|^{u_{k(n)}}$$
$$= \frac{1}{n}\ \|\rho, \eta\|^{u_{k(n)}}$$
$$\le \frac{1}{n}\ A\,[\,\|\rho, \eta\|^{L(u)}\,] \ \to 0 \text{ as } n \to \infty$$

and

$$\|\alpha_k \xi_k, \eta\|^{u_k} = 0, \text{ for } k \ne k(n),\ n \ge 1.$$

This shows that $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$. But on the other hand, let us choose $\eta \in S$ such that $\|\rho, \eta\| = 1$. Then for $k = k(n)$, $n \ge 1$, in view of (3.5) and (3.6) ,we have

$$\|\alpha_k \xi_k, \eta\|^{v_k} = \|\alpha_{k(n)} \xi_{k(n)}, \eta\|^{v_{k(n)}}$$
$$= \|n^{-1/u_{k(n)}}\rho, \eta\|^{v_{k(n)}}$$
$$\ge \frac{1}{n^{1/n}}\ \|\rho, \eta\|^{v_{k(n)}}$$
$$\ge \frac{1}{n^{1/2}}$$

This shows that $\bar{\xi} \notin c_0(S, \bar{\alpha}, \bar{v}, \|.,.\|)$, which contradicts our assumption.

For the sufficiency of the condition, suppose that $\liminf_k \frac{v_k}{u_k} > 0$. Then there exists a $m > 0$ such that $v_k > m\, u_k$ for all sufficiently large values of k.

Let $\bar{\xi} = (\xi_k) \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$. Then for each $\eta \in S$

$$\|\alpha_k \xi_k, \eta\|^{u_k} \to 0 \text{ as } k \to \infty.$$

Hence for a given $\varepsilon > 0$, if we choose $0 < \delta < 1$ satisfying $\delta^m < \varepsilon$ satisfying

$$||\alpha_k \xi_k, \eta||^{u_k} < \delta < 1$$

for each $\eta \in S$ and for all sufficiently large values of k.Thus

$$||\alpha_k \xi_k, \eta||^{v_k} \leq [||\alpha_k \xi_k, \eta||^{u_k}]^m \leq \delta^m < \varepsilon,$$

for each $\eta \in S$ and for all sufficiently large values of k and consequently $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{v}, ||.,.||)$. Hence

$$c_0(S, \bar{\alpha}, \bar{u}, ||.,.||) \subset c_0(S, \bar{\alpha}, \bar{v}, ||.,.||).$$

This completes the proof of the theorem.

Theorem 3.6: For any $\bar{\alpha} = (\alpha_k)$, $c_0(S, \bar{\alpha}, \bar{v}, ||.,.||) \subset c_0(S, \bar{\alpha}, \bar{u}, ||.,.||)$ if and only if $\limsup_k \frac{v_k}{u_k} < \infty$.

Proof:

For the necessity, suppose that the inclusion holds but $\limsup_k \frac{v_k}{u_k} = \infty$. Then there exists a sequence $(k(n))$ of positive integers such that

$$1 \leq k(n) < k(n+1), \; n \geq 1$$

for which

$$v_{k(n)} > n\, u_{k(n)}, \text{for all } n \geq 1. \qquad ...(3.7)$$

Corresponding to $\rho \in S$ and $\rho \neq \theta$, we define a sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1}\, n^{-1/v_{k(n)}} \rho, \text{ for } k = k(n), \; n \geq 1 \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \qquad ...(3.8)$$

Then for each $\eta \in S$, $k = k(n)$, $n \geq 1$, we have

$$||\alpha_k \xi_k, \eta||^{v_k} = ||n^{-1/v_{k(n)}} \rho, \eta||^{v_{k(n)}} = \frac{1}{n} ||\rho, \eta||^{v_{k(n)}} \leq \frac{1}{n} A[||\rho, \eta||^{L(v)}] \to 0 \text{ as } n \to \infty$$

and

$$||\alpha_k \xi_k, \eta||^{v_k} = 0, \text{ for } k \neq k(n), \; n \geq 1.$$

This shows that $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{v}, ||.,.||)$. But on the other hand, let us choose $\eta \in S$ such that $||\rho, \eta|| = 1$. Then for $k = k(n)$, $n \geq 1$, in view of (3.7) and (3.8) ,we have

$$||\alpha_k \xi_k, \eta||^{u_k} = ||n^{-1/v_{k(n)}} \rho, \eta||^{u_{k(n)}} \geq \frac{1}{n^{1/n}} ||\rho, \eta||^{u_{k(n)}} \geq \frac{1}{n^{1/2}}$$

This shows that $\bar{\xi} \notin c_0(S, \bar{\alpha}, \bar{u}, ||.,.||)$, a contradiction.

For the sufficiency of the condition, assume that $\limsup_k \frac{v_k}{u_k} < \infty$. Hence there exists $d > 0$ such that $v_k < d\, u_k$ for all sufficiently large values of k.

Let $\bar{\xi} = (\xi_k) \in c_0(S, \bar{\alpha}, \bar{v}, ||.,.||)$. Then for each $\eta \in S$

$$||\alpha_k \xi_k, \eta||^{v_k} \to 0 \text{ as } k \to \infty.$$

Hence for a given $\varepsilon > 0$, if we choose $0 < \delta < 1$ satisfying $\delta^{1/d} < \varepsilon$ satisfying

$$||\alpha_k \xi_k, \eta||^{v_k} < \delta < 1$$

for each $\eta \in S$ and for all sufficiently large values of k.Thus

$$||\alpha_k \xi_k, \eta||^{u_k} \leq [||\alpha_k \xi_k, \eta||^{v_k}]^{1/d}$$

$$\leq \delta^{1/d} < \varepsilon,$$

for each $\eta \in S$ and for all sufficiently large values of k and consequently $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$. Hence

$$c_0(S, \bar{\alpha}, \bar{v}, \|.,.\|) \subset c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|).$$

This completes the proof.

On combining the Theorems 3.5 and 3.6, one obtain

Theorem 3.7: For any $\bar{\alpha} = (\alpha_k)$, $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) = c_0(S, \bar{\alpha}, \bar{v}, \|.,.\|)$ if and only if

$$0 < \liminf_k \frac{v_k}{u_k} \leq \limsup_k \frac{v_k}{u_k} < \infty.$$

Corollary 3.8: For any $\bar{\alpha} = (\alpha_k)$,

(i) $c_0(S, \bar{\alpha}, \|.,.\|) \subset c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ **if and only if** $\liminf_k u_k > 0$;

(ii) $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\alpha}, \|.,.\|)$ **if and only if** $\limsup_k u_k < \infty$; **and**

(iii) $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) = c_0(S, \bar{\alpha}, \|.,.\|)$ **if and only if**

$$0 < \liminf_k u_k \leq \limsup_k u_k < \infty.$$

Proof:

Proof follows by taking $u_k = 1$ for all k and replacing $\bar{v}$ by $\bar{u}$ in Theorems 3.5 , 3.6 and 3.7.

Theorem 3.9: For any $\bar{\alpha} = (\alpha_k)$, $\bar{\beta} = (\beta_k)$, $\bar{u} = (u_k)$ and $\bar{v} = (v_k)$,

$c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|) \subset c_0(S, \bar{\beta}, \bar{v}, \|.,.\|)$ if and only if

(i) $\liminf_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} > 0$; **and**

(ii) $\liminf_k \frac{v_k}{u_k} > 0$.

Proof:

Proof of the theorem follows immediately from the Theorems 3.1 and 3.5.

In the following example, $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ may strictly be contained in $c_0(S, \bar{\beta}, \bar{v}, \|.,.\|)$ in spite of the satisfaction of the conditions (i) and (ii) of Theorem 3.9.

Example 3.10:

Let $(S, \|.,.\|)$ be a 2- normed space and consider a sequence $\bar{\xi} = (\xi_k)$ defined by

$$\xi_k = k^{-k}\rho, \text{ if } k = 1, 2, 3, \ldots,$$

where $\rho \in S$ and $\rho \neq \theta$.

Further, let $u_k = k^{-1}$, if k is odd integer, $u_k = k^{-2}$, if k is even integer, $v_k = k^{-1}$ for all values of k, $\alpha_k = 3^k$, $\beta_k = 2^k$ for all values of k.

Then

$$\left|\frac{\alpha_k}{\beta_k}\right|^{u_k} = \frac{3}{2} \text{ or } \left(\frac{3}{2}\right)^{1/k} \text{according as } k \text{ is odd or even integer}$$

and hence

$$\liminf_k \left|\frac{\alpha_k}{\beta_k}\right|^{u_k} > 0.$$

Further, $\frac{v_k}{u_k} = 1$, if k is odd integer, $\frac{v_k}{u_k} = k$, if k is even integer.

Therefore $\liminf_k \frac{v_k}{u_k} > 0$. Hence the conditions (i) and (ii) of Theorem 3.9 are satisfied.

Now for each $\eta \in S$, we have

$$\|\beta_k\, \xi_k\,,\, \eta\|^{v_k} = \|2^k\, k^{-k}\, \rho\,,\, \eta\|^{1/k}$$
$$\leq \frac{1}{k}\;\; 2\, \|\, \rho\,,\, \eta\|^{1/k}$$
$$\leq \frac{1}{k}\;\; 2A\, [\, \|\, \rho,\, \eta\, \|\,] \;\to 0 \;\text{ as } k \to \infty,$$

and it shows that $\bar{\xi} \in c_0\,(\,S,\; \bar{\beta},\; \bar{v}, \|.\,,.\|\,)$.
But on the other hand, let us choose $\eta \in S$ such that $\|\,\rho, \eta\,\| = 1$. Then for each even integer k, we have

$$\|\, \alpha_k\, \xi_k\,,\, \eta\|^{u_k} = \|3^k\, k^{-k}\, \rho\,,\, \eta\|^{1/k^2}$$
$$= (3/k)^{1/k}\, \|\, \rho\,,\, \eta\|^{1/k^2}$$
$$> \frac{1}{2}.$$

This implies that $\bar{\xi} \notin c_0\,(\,S,\; \bar{\alpha}, \bar{u}\,, \|.\,,.\|\,)$. Thus the containment of $c_0\,(\,S,\; \bar{\alpha}, \bar{u}\,, \|.\,,.\|\,)$ in $c_0\,(\,S, \bar{\beta}, \bar{v}, \|.\,,.\|\,)$ is strict inspite of the satisfaction of the conditions (i) and (ii) of the Theorem 3.9.

4. Linear Topological Structures of $c_0\,(\,S,\; \bar{\alpha}, \bar{u}\,, \|.\,,.\|\,)$

In this section, we shall investigate some results that characterize the linear topological structure of the class $c_0\,(\,S,\; \bar{\alpha}, \bar{u}\,, \|.\,,.\|\,)$ by endowing it with suitable natural paranorm. Throughout we take coordinatewise operations of sequences over the field ***C*** of complex numbers i.e.,for $\bar{\xi} = (\xi_k)$ and $\bar{\eta} = (\eta_k)$ and scalar γ,

$$\bar{\xi} + \bar{\eta} = (\xi_k + \eta_k)$$

and

$$\gamma\, \bar{\xi} = (\gamma\, \xi_k)$$

and we see below that each of these classes forms a complete paranormed linear space over ***C.*** Moreover, we use frequently

$$|\, \xi + \eta\, |^{u_k} \leq S\, \{|\xi|^{u_k} + |\eta|^{u_k}\},$$

where $\xi\,, \eta \in \mathbf{C}$, $0 < u_k \leq \sup_k u_k = L < \infty$ and $A(\alpha) = \max\,(1, |\alpha|\,)$.

Theorem 4.1: The space $c_0\,(\,S,\; \bar{\alpha}, \bar{u}\,, \|.\,,.\|\,)$ forms a linear space over *C* if and only if $sup_k\, u_k$ is bounded above .

Proof:

For the necessity of the conditions**,** suppose that $c_0\,(\,S, \bar{\alpha}, \bar{u}\,, \|.\,,.\|\,)$ is a linear space over ***C*** but $sup_k\; u_k = \infty$. Then there exists a sequence $(k(n))$ of positive integers satisfying

$$1 \leq\; k(n) < k(n+1),\; n \geq 1$$

for which

$$u_{k(n)} \;>\; n \;\;\text{,for each } n \geq 1 \qquad \ldots .(4.1)$$

Now, corresponding to $\rho \in S$ and $\rho \neq \theta$, we define the sequence $\bar{\xi} = (\xi_k)$ by

$$\xi_k = \begin{cases} \alpha_{k(n)}^{-1}\; n^{-2/u_{k(n)}}\, \rho, \text{ if } k = k(n)\;,\; n \geq 1 \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \qquad \ldots .(4.2)$$

Then for $k = k(n)$, $n \geq 1$, we have

$$\|\alpha_k\, \xi_k,\, \eta\|^{u_k} = \|n^{-2/u_{k(n)}}\, \rho,\, \eta\,\|^{u_{k(n)}})$$
$$\leq \frac{\|\rho,\, \eta\,\|^{u_{k(n)}}}{n^2} \;\to 0 \text{ as } n \to \infty,$$

and

$$\|\alpha_k\, \xi_k,\, \eta\|^{u_k} = 0, \text{ for } k \neq k(n)\,,\, n \geq 1$$

showing that $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$.

But on the other hand, let us choose $\eta \in S$ such that $\|\rho, \eta\| = 1$. Then for such η and scalar $\gamma = 4$, for $k = k(n)$, $n \geq 1$, in view of (4.1) and (4.2), we have

$$\begin{aligned} \|\gamma\, \alpha_k\, \xi_k, \eta\|^{u_k} &= \|\alpha_{k(n)}\, \gamma\, \xi_{k(n)}, \eta\|^{u_{k(n)}} \\ &= \|4\, n^{-2/u_{k(n)}}\, \rho, \eta\|^{u_{k(n)}} \\ &= \frac{4^{u_{k(n)}}}{n^2} \\ &\geq \sup_n \frac{4^n}{n^2} \geq 1\,. \end{aligned}$$

This shows that $\gamma \bar{\xi} \notin c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$, a contradiction.

For the sufficiency of the condition, assume that $\sup_k u_k < \infty$. Let $\bar{\xi}, \bar{\eta} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ and γ, $\rho \in \boldsymbol{C}$. Then for each $\eta \in S$, we have

$$\|\alpha_k\, \xi_k, \eta\|^{u_k} \to 0$$

and

$$\|\alpha_k\, \eta_k, \eta\|^{u_k} \to 0, \text{ as } k \to \infty.$$

We now setting $S = \max(1, 2^{L-1})$, then we have

$$\begin{aligned} \|\alpha_k(\gamma\xi_k + \rho\eta_k), \eta\|^{u_k} &= (\|\gamma\alpha_k\, \xi_k, \eta\| + \|\rho\, \alpha_k\, \eta_k, \eta\|)^{u_k} \\ &\leq (S\,\|\gamma\, \alpha_k\, \xi_k, \eta\|^{u_k} + S\,\|\rho\, \alpha_k\, \eta_k, \eta\|^{u_k}) \\ &= S\, |\gamma|^{u_k}\, \|\alpha_k\, \xi_k, \eta\|^{u_k} + S\, |\rho|^{u_k}\, \|\alpha_k\, \eta_k, \eta\|^{u_k} \end{aligned}$$

which tends to 0 as $k \to \infty$, for each $\eta \in S$ and hence $\alpha\bar{\xi} + \rho\,\bar{\eta} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$.

This implies that $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ forms a linear space over $\boldsymbol{C}$. This completes the proof.

Let $\bar{u} \in \ell_\infty$ i.e., $\sup_k u_k < \infty$, $M = \max(1, \sup_k u_k)$ and consider a real valued function T on $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ defined by

$$T(\bar{x}) = \{\sup_k \|\alpha_k\, \xi_k, \eta\|^{u_k/M}, \text{ for each } \eta \in S\}, \text{for } \bar{\xi} \in \ell_\infty(S, \bar{\alpha}, \bar{u}, \|.,.\|). \qquad \dots (4.3)$$

We prove below that $c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$ with respect to T forms a paranormed space.

Theorem: 4.2 ($c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$, T) forms a total paranormed space.

Proof:

Since $T(\bar{\xi}) \geq 0$; $T(\bar{\xi}) = 0$ if and only if $\bar{\xi} = \bar{\theta}$; $T(-\bar{\xi}) = T(\bar{\xi})$; and

$$T(\bar{\xi} + \bar{\eta}) \leq T(\bar{\xi}) + T(\bar{\eta})$$

easily follow. So PN_1, PN_2 and PN_3 are obvious.

For PN_4 i.e., the continuity of scalar multiplication, it suffices to show that

(a) if $\bar{\xi}^{(n)} \to \bar{\theta}$ in T and $\gamma_n \to \gamma$ as $n \to \infty$, then $\gamma_n\, \bar{\xi}^{(n)} \to \bar{\theta}$ in T;

and

(b) if $\gamma_n \to 0$ and $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$, then $\gamma_n\, \bar{\xi} \to \bar{\theta}$ in T.

Now (a) is easily proved if we suppose $|\gamma_n| \leq L$ for all $n \geq 1$ and in view of (4.3), we have

$$\begin{aligned} T(\gamma_n\, \bar{\xi}^{(n)}) &= \{\sup_k \|\gamma_n\, \alpha_k\, \xi_k^{(n)}, \eta\|^{u_k/M}, \text{ for each } \eta \in S\} \\ &\leq \{\sup_k |\gamma_n|^{u_k/M} \sup_k \|\alpha_k\, \xi_k^{(n)}, \eta\|^{u_k/M}, \text{ for each } \eta \in S\} \end{aligned}$$

$$= A(L) \{ \sup_k \| \alpha_k \xi_k^{(n)}, \eta \|^{u_k/M}, \text{ for each } \eta \in S \}$$

$$\leq A(L)\, T(\bar{\xi}^{(n)}),$$

implies that

$$T(\gamma_n \bar{\xi}^{(n)}) \to 0, \text{ as } T(\bar{\xi}^{(n)}) \to 0 \text{ as } n \to \infty.$$

To prove (b), let $\bar{\xi} \in c_0(S, \bar{\alpha}, \bar{u}, \|.,.\|)$, $|\gamma_n| < 1$ for all $n \geq N_1$ and $\varepsilon > 0$. Then there exists a positive integer K such that

$$\|\alpha_k \xi_k, \eta\|^{u_k} < \varepsilon^M$$

for all $k \geq K$ and for each $\eta \in S$ and hence for all $k \geq K$ and $n \geq N_1$, we have

$$\| \gamma_n \alpha_k \xi_k, \eta \|^{u_k} = |\gamma_n|^{u_k} \| \alpha_k \xi_k, \eta \|^{u_k} < \varepsilon^M, \text{ for each } \eta \in S.$$

Now choose N_2 such that for all $k = 1, 2, \ldots, K-1$ and $n \geq N_2$,

$$\| \gamma_n \alpha_k \xi_k, \eta \|^{u_k} = |\gamma_n|^{u_k} \| \alpha_k \xi_k, \eta \|^{u_k} < \varepsilon^M, \text{ for each } \eta \in S.$$

Thus, $T(\gamma_n \bar{\xi}) \leq \varepsilon$, for $n \geq N = \max(N_1, N_2)$, which proves (b).The totality of T is obvious.

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