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On The Eneström-Kakeya Theorem

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Abstract

In this paper we present some interesting generalizations of Eneström-Kakeya type results concerning the location of zeros of a polynomial in the complex plane. We relax the hypothesis and put less restrictive conditions on the coefficients of the polynomial, and thereby generalize some classical results.

Keywords: polynomials, zeros, Eneström-Kakeya theorem

1. Introduction

The following result known as Eneström-Kakeya Theorem (for Reference see [1-3]) is well-known in the theory of distribution of zeros of polynomials.

Theorem A. If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

(1) $$a_n \ge a_{n-1} \ge .... \ge a_1 \ge a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \le 1$.

Joyal, Labelle and Rahman [4] dropped the restriction on the hypothesis that all the coefficients be positive and proved the following:

Theorem B. If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge .... \ge a_1 \ge a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \le \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

More recently Aziz and Zargar [5] relaxed the hypothesis of Eneström Kakeya theorem and proved some interesting extensions of Theorem B. In fact they proved

THEOREM C. If $P(z)=a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, is a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \geq a_{n-1} \geq ... \geq a_2 \geq a_1 \geq a_0,$$

then all the zeros of P(z) lie in

$$|z+k-1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

More recently, Shah and Liman [6] while assuming the coefficients of the polynomials to be complex numbers among other things proved the following generalization of Theorem C.

Theorem D. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients. If $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0,1,2,...,n$, $a_n \neq 0$ such that for some $k \geq 1$

$$k\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq ... \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq .... \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|}$$

In this paper, we prove some Enström-Kakeya type results which besides generalizing some earlier results proved in this direction, include Theorem B, Theorem C and Theorem D as special cases. We start by proving the following:

2. Theorems and Proofs

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n**.** If $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j$ for $j = 0,1,2,...,n$, $a_n \neq 0$ and if for some positive integer $\lambda \leq n$ and $k \geq 1$

$$k^{n-\lambda+1}\alpha_n \geq k^{n-\lambda}\alpha_{n-1} \geq k^{n-\lambda-1}\alpha_{n-2} \geq ... \geq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ... \geq \alpha_1 \geq \alpha_0$$

then all the zeros of P(z) lie in

$$|z+k-1| \leq \frac{\left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{j=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + 2\sum_{j=0}^{n} |\beta_j| \right\}}{|\alpha_n|}.$$

If we take $\lambda = n$ we obtain the following **:**

Corollary 1.1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re} a_j = \alpha_j$**,** $\operatorname{Im} a_j = \beta_j$ for $j = 0,1,2,...,n$, $a_n \neq 0$ and if for some $k \geq 1$

$$k\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq ... \geq \alpha_1 \geq \alpha_0$$

then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

For $\lambda = n-1$ and assuming all coefficients real and positive in Theorem 1, we obtain the following

Corollary 1.2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and if for some $k \ge 1$

$$k^2 a_n \ge k a_{n-1} \ge a_{n-2} \ge ... \ge a_1 \ge a_0 > 0$$

then all the zeros of P(z) lie in

$$|z+k-1| \le k + 2(k-1)\frac{a_{n-1}}{a_n}$$

or equivalently

$$|z+k-1| \le k(2k-1)$$

Remark 1.1: Theorem C is a special case of Theorem 1, if we take all coefficients real and $\lambda = n$. Further for all coefficients real, $\lambda = n$ and $k = 1$ we obtain Theorem B.

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0,1,2,...,n$, $a_n \ne 0$ and if for some positive integer $\lambda \le n$ and $k \ge 1$

$$k^{n-\lambda+1}\beta_n \ge k^{n-\lambda}\beta_{n-1} \ge k^{n-\lambda-1}\beta_{n-2} \ge ... \ge k\beta_\lambda \ge \beta_{\lambda-1} \ge ... \ge \beta_1 \ge \beta_0$$

then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{\left\{\beta_n - \beta_0 + |\beta_0| + (k-1)\left(\sum_{i=\lambda}^{n}(\beta_j + |\beta_j|) - |\beta_n|\right) + 2\sum_{j=0}^{n}|\alpha_j|\right\}}{|\beta_n|}$$

Next as a generalization of Therorem D, we prove

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0,1,2,...,n$, $a_n \ne 0$ and if for some positive integer $\lambda \le n$ and $k \ge 1$

$$k^{n-\lambda+1}\alpha_n \ge k^{n-\lambda}\alpha_{n-1} \ge k^{n-\lambda-1}\alpha_{n-2} \ge ... \ge k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ... \ge \alpha_1 \ge \alpha_0$$

$$\beta_n \ge \beta_{n-1} \ge ... \ge \beta_1 \ge \beta_0 > 0$$

then all the zeros of P(z) lie in

$$\left|z + \frac{\alpha_n}{a_n}(k-1)\right| \le \frac{\left\{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{i=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right) + \beta_n\right\}}{|a_n|}.$$

Remark 3.1: If we take $\lambda = n$ in above we obtain Theorem D.

Theorem 4: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive integer $\lambda \le n, k \ge 1$ and $\mu \ge 1$

$$k^{n-\lambda+1}\alpha_n \ge k^{n-\lambda}\alpha_{n-1} \ge k^{n-\lambda-1}\alpha_{n-2} \ge \ldots \ge k\alpha_\lambda \ge \alpha_{\lambda-1} \ge \ldots \ge \alpha_1 \ge \alpha_0$$

$$\mu\beta_n \ge \beta_{n-1} \ge \ldots \ge \beta_1 \ge \beta_0 > 0$$

then all the zeros of P(z) lie in

$$\left| z + \frac{k\alpha_n + i\mu\beta_n}{a_n} - 1 \right| \le \frac{\left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{i=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + \mu\beta_n \right\}}{|a_n|}$$

If we take $\lambda = n$ in above we obtain the following:

Corollary 4.1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive integer $\lambda \le n$ $, k \ge 1$ and $\mu \ge 1$

$$k\alpha_n \ge \alpha_{n-1} \ge \ldots \ge \alpha_1 \ge \alpha_0$$

$$\mu\beta_n \ge \beta_{n-1} \ge \ldots \ge \beta_1 \ge \beta_0 > 0$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{k\alpha_n + i\mu\beta_n}{a_n} - 1 \right| \le \frac{\{k\alpha_n - \alpha_0 + |\alpha_0| + \mu\beta_n\}}{|a_n|}$$

Theorem 5: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive integer $\lambda \le n$ and $k \ge 1$

$$k^{n-\lambda+1}|a_n| \ge k^{n-\lambda}|a_{n-1}| \ge k^{n-\lambda-1}|a_{n-2}| \ge \ldots \ge k|a_\lambda| \ge |a_{\lambda-1}| \ge \ldots \ge |a_1| \ge |a_0|$$

and for some real β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \; j = 0,1,2,\ldots$$

then all the zeros of lie in

$$|z + k - 1| \le k(\cos\alpha + \sin\alpha) + \frac{1}{|a_n|}\{(k-1)\cos\alpha + (k+1)\sin\alpha\} \sum_{i=\lambda+1}^{n} |a_j| + (k-1)\sum_{i=\lambda}^{n-1} | a_i | + \frac{2\sin\alpha}{|a_n|} \sum_{j=\lambda-1}^{1} |a_j|.$$

Proof of Theorem 1: Consider a polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0 \\ &= \left\{ -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 \right\} \\ &\quad + i\left\{ -\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0 \right\} \end{aligned}$$

$$= \{-\alpha_n z^n(z+k-1) + (k\alpha_n - \alpha_{n-1})z^n + (k\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots + (k\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}$$
$$+ (k\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 - (k-1)(\alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2}$$
$$+ \ldots + \alpha_{\lambda+1}z^{\lambda+1} + \alpha_\lambda z^\lambda)\} + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0\}$$
$$= \{-\alpha_n z^n(z+k-1) + (k\alpha_n - \alpha_{n-1})z^n + (k\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots + (k\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1}$$
$$+ (k\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 - (k-1)(\alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2}$$
$$+ \ldots + \alpha_{\lambda+1}z^{\lambda+1} + \alpha_\lambda z^\lambda)\} - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j\}$$

$$= z^n\left\{-\alpha_n(z+k-1) + (k\alpha_n - \alpha_{n-1}) + (k\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \ldots + (k\alpha_{\lambda+1} - \alpha_\lambda)\frac{1}{z^{n-\lambda-1}} + (k\alpha_\lambda - \alpha_{\lambda-1})\frac{1}{z^{n-\lambda}} + \right.$$
$$\left. + \ldots + (\alpha_1 - \alpha_0)\frac{1}{z^{n-1}} + \alpha_0\frac{1}{z^n} - (k-1)\left\{\frac{\alpha_{n-1}}{z} + \frac{\alpha_{n-2}}{z^2} + \ldots + \frac{\alpha_\lambda}{z^{n-\lambda}}\right\} - i\beta_n z + i\frac{\beta_0}{z^n} + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})\frac{1}{z^{n-j}}\right\}$$

Let $|z| > 1$ so that $\frac{1}{|z|^{n-j}} < 1$, $j = 0,1,2,\ldots$

$$F(z) \geq |z|^n\left\{|\alpha_n||z+k-1| - \left[|k\alpha_n - \alpha_{n-1}| + \frac{|k\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \ldots + \frac{|k\alpha_{\lambda+1} - \alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|k\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} + \right.\right.$$
$$+ \ldots + \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + |k-1|\left(\frac{|\alpha_{n-1}|}{|z|} + \frac{|\alpha_{n-2}|}{|z|^2} + \ldots + \frac{|\alpha_{\lambda+1}|}{|z|^{n-\lambda-1}} + \frac{|\alpha_\lambda|}{|z|^{n-\lambda}}\right)$$
$$\left.\left. + |\beta_n||z| + \frac{|\beta_0|}{|z|^n} + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)\frac{1}{|z|^{n-j}}\right]\right\}$$
$$> |z|^n\{|\alpha_n||z+k-1| - [|k\alpha_n - \alpha_{n-1}| + |k\alpha_{n-1} - \alpha_{n-2}| + \ldots + |k\alpha_{\lambda+1} - \alpha_\lambda| + |k\alpha_\lambda - \alpha_{\lambda-1}| +$$
$$+ \ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + |k-1|\sum_{j=\lambda}^{n-1}|\alpha_j| + |\beta_n| + |\beta_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)]\}$$

$$= |z|^n\{|\alpha_n||z+k-1| - [|k\alpha_n - \alpha_{n-1}| + |k\alpha_{n-1} - \alpha_{n-2}| + \ldots + |k\alpha_{\lambda+1} - \alpha_\lambda| + |k\alpha_\lambda - \alpha_{\lambda-1}| +$$
$$+ \ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + |k-1|\sum_{j=\lambda}^{n-1}|\alpha_j| + |\beta_n| + |\beta_0| + \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)]\}$$

$$= |z|^n\left\{|\alpha_n||z+k-1| - \left[\left\{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right)\right\} + 2\sum_{j=0}^{n}|\beta_j|\right]\right\}$$

$> 0,$ if

$$|z+k-1| > \frac{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

This shows that those zeros of $F(z)$ for which $|z| > 1$ lie in

$$|z+k-1| \le \frac{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

But those zeros of $F(z)$ for which $|z| \le 1$ already satisfy the above equation. We conclude that all the zeros of $F(z)$ and hence of $P(z)$ lie in the disc

$$|z+k-1| \le \frac{\alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{i=\lambda}^{n}(\alpha_j + |\alpha_j|) - |\alpha_n|\right) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

This completes the proof.

Proof of Theorem 2: The proof follows on the same lines of Theorem 1.

Proof of Theorem 3: Consider a polynomial

$$
\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0 \\
&= \{-a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0\} \\
&\qquad + i\{(\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{-a_n z^{n+1} - \alpha_n z^n(k-1) + (k\alpha_n - \alpha_{n-1})z^n + (k\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots + (k\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} \\
&\qquad + (k\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 - (k-1)(\alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} \\
&\qquad + \ldots + \alpha_{\lambda+1}z^{\lambda+1} + \alpha_\lambda z^\lambda)\} + i\{(\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}
$$

$$
\begin{aligned}
&= z^n \Bigg\{ -a_n z - \alpha_n(k-1) + (k\alpha_n - \alpha_{n-1}) + (k\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \ldots + (k\alpha_{\lambda+1} - \alpha_\lambda)\frac{1}{z^{n-\lambda-1}} \\
&\qquad + (k\alpha_\lambda - \alpha_{\lambda-1})\frac{1}{z^{n-\lambda}} + \ldots + (\alpha_1 - \alpha_0)\frac{1}{z^{n-1}} + \alpha_0\frac{1}{z^n} - (k-1)\left\{\frac{\alpha_{n-1}}{z} + \frac{\alpha_{n-2}}{z^2} + \ldots + \frac{\alpha_\lambda}{z^{n-\lambda}}\right\} + \\
&\qquad + i\left\{(\beta_n - \beta_{n-1}) + \ldots + (\beta_1 - \beta_0)\frac{1}{z^{n-1}} + \beta_0\frac{1}{z^n}\right\}\Bigg\}
\end{aligned}
$$

$$= z^n \left\{ -a_n z - \alpha_n (k-1) + (k\alpha_n - \alpha_{n-1}) + (k\alpha_{n-1} - \alpha_{n-2}) \frac{1}{z} + ... + (k\alpha_{\lambda+1} - \alpha_\lambda) \frac{1}{z^{n-\lambda-1}} \right.$$

$$+ (k\alpha_\lambda - \alpha_{\lambda-1}) \frac{1}{z^{n-\lambda}} + ... + (\alpha_1 - \alpha_0) \frac{1}{z^{n-1}} + \alpha_0 \frac{1}{z^n} - (k-1) \left\{ \frac{\alpha_{n-1}}{z} + \frac{\alpha_{n-2}}{z^2} + ... + \frac{\alpha_\lambda}{z^{n-\lambda}} \right\}$$

$$\left. + i \left\{ (\beta_n - \beta_{n-1}) + ... + (\beta_1 - \beta_0) \frac{1}{z^{n-1}} + \beta_0 \frac{1}{z^n} \right\} \right\}$$

Let $|z| > 1$ so that $\frac{1}{|z|^{n-j}} < 1$, $j = 0,1,2,...$

$$|F(z)| \geq |z|^n \left\{ |a_n z + (k-1)\alpha_n| - \left\{ |k\alpha_n - \alpha_{n-1}| + \frac{|k\alpha_{n-1} - \alpha_{n-2}|}{|z|} + ... + \frac{|k\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} + \right. \right.$$

$$+ \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + (k-1) \left(\frac{|\alpha_{n-1}|}{|z|} + \frac{|\alpha_{n-2}|}{|z|^2} + ... + \frac{|\alpha_{n-2}|}{|z|^2} \right)$$

$$\left. \left. + i \left(|\beta_n - \beta_{n-1}| + ... + \frac{|\beta_1 - \beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \right) \right\} \right\}$$

$$= |z|^n \left[|a_n| \left| z + (k-1) \frac{\alpha_n}{a_n} \right| - \left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1) \left(\sum_{i=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + \beta_n \right\} \right]$$

$$> 0,$$

if

$$\left| z + (k-1) \frac{\alpha_n}{a_n} \right| > \frac{\left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1) \left(\sum_{i=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + \beta_n \right\}}{|a_n|}$$

This shows that the zeros of $F(z)$, for which $|z| > 1$ lie in

$$\left| z + (k-1) \frac{\alpha_n}{a_n} \right| \leq \frac{\left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1) \left(\sum_{i=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + \beta_n \right\}}{|a_n|}.$$

But those zeros of $F(z)$ for which $|z| \leq 1$ already satisfy the above equation. We conclude that all the zeros of $F(z)$ and hence of $P(z)$ lie in the disc

$$\left| z + (k-1) \frac{\alpha_n}{a_n} \right| \leq \frac{\left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1) \left(\sum_{i=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + \beta_n \right\}}{|a_n|}$$

Proof of Theorem 4: Consider a polynomial

$$
\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0 \\
&= \{-a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0\} \\
&\quad + i\{+(\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{-a_n z^{n+1} - \alpha_n z^n(k-1) + (k\alpha_n - \alpha_{n-1})z^n + (k\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots + (k\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} \\
&\quad + (k\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots + (\alpha_1 - \alpha_0)z + \alpha_0 - (k-1)(\alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} + \ldots + \alpha_{\lambda+1}z^{\lambda+1} \\
&\quad + \alpha_\lambda z^\lambda)\} + i\{(\mu\beta_n + \beta_n - \mu\beta_n - \beta_{n-1})z^n + \ldots + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}
$$

$$
\begin{aligned}
&= z^n \left\{ (-a_n z - \alpha_n k + \alpha_n - i\mu\beta_n + i\beta_n) + (k\alpha_n - \alpha_{n-1}) + (k\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \ldots \right. \\
&\quad + (k\alpha_\lambda - \alpha_{\lambda-1})\frac{1}{z^{n-\lambda}} + + (\alpha_1 - \alpha_0)\frac{1}{z^{n-1}} + \alpha_0\frac{1}{z^n} - (k-1)\left\{\frac{\alpha_{n-1}}{z} + \frac{\alpha_{n-2}}{z^2} + \ldots + \frac{\alpha_\lambda}{z^{n-\lambda}}\right\} \\
&\quad \left. + i\left\{(\mu\beta_n - \beta_{n-1}) + \ldots + (\beta_1 - \beta_0)\frac{1}{z^{n-1}} + \beta_0\frac{1}{z^n}\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= z^n \left[(-a_n z - \alpha_n k - i\mu\beta_n + a_n) + \left\{ \left((k\alpha_n - \alpha_{n-1}) + (k\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \ldots \right.\right.\right. \\
&\quad \left. + (k\alpha_\lambda - \alpha_{\lambda-1})\frac{1}{z^{n-\lambda}} + \ldots + (\alpha_1 - \alpha_0)\frac{1}{z^{n-1}} + \alpha_0\frac{1}{z^n}\right) - (k-1)\left(\frac{\alpha_{n-1}}{z} + \frac{\alpha_{n-2}}{z^2} + \ldots \frac{\alpha_\lambda}{z^{n-\lambda}}\right) \\
&\quad \left. + i\left((\mu\beta_n - \beta_{n-1}) + \ldots + (\beta_1 - \beta_0)\frac{1}{z^{n-1}} + \beta_0\frac{1}{z^n}\right)\right\}
\end{aligned}
$$

Let $|z| > 1$ so that $\dfrac{1}{|z|^{n-j}} < 1$, $j = 0,1,2,\ldots$

$$
\begin{aligned}
|F(z)| &\geq |z|^n \left[|a_n| \left| z + \frac{k\alpha_n + i\mu\beta_n}{a_n} - 1 \right| - \{ |k\alpha_n - \alpha_{n-1}| + |k\alpha_{n-1} - \alpha_{n-2}| + \ldots + |k\alpha_\lambda - \alpha_{\lambda-1}| + \right. \\
&\quad \ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + (k-1)(|\alpha_{n-1}| + |\alpha_{n-2}| + \ldots + |\alpha_\lambda|) \\
&\quad \left. + |\mu\beta_n - \beta_{n-1}| + \ldots + |\beta_1 - \beta_0| + |\beta_0| \} \right] \\
&= |z|^n \left[|a_n| \left| z + \frac{k\alpha_n + i\mu\beta_n}{a_n} - 1 \right| - \left\{ \alpha_n - \alpha_0 + |\alpha_0| + (k-1)\left(\sum_{i=\lambda}^{n} (\alpha_j + |\alpha_j|) - |\alpha_n| \right) + \mu\beta_n \right\} \right]
\end{aligned}
$$

$$> 0,$$

$$\left|z+\frac{k\alpha_n+i\mu\beta_n}{a_n}-1\right|>\frac{\left\{\alpha_n-\alpha_0+|\alpha_0|+(k-1)\left(\sum_{i=\lambda}^{n}(\alpha_j+|\alpha_j|)-|\alpha_n|\right)+\mu\beta_n\right\}}{|a_n|}$$

This shows that the zeros of $F(z)$, for which $|z|>1$ lie in

$$\left|z+\frac{k\alpha_n+i\mu\beta_n}{a_n}-1\right|\le\frac{\left\{\alpha_n-\alpha_0+|\alpha_0|+(k-1)\left(\sum_{i=\lambda}^{n}(\alpha_j+|\alpha_j|)-|\alpha_n|\right)+\mu\beta_n\right\}}{|a_n|}.$$

But those zeros of $F(z)$ for which $|z|\le 1$ already satisfy the above equation. We conclude that all the zeros of $F(z)$ and hence of $P(z)$ lie in the disc.

$$\left|z+\frac{k\alpha_n+i\mu\beta_n}{a_n}-1\right|\le\frac{\left\{\alpha_n-\alpha_0+|\alpha_0|+(k-1)\left(\sum_{i=\lambda}^{n}(\alpha_j+|\alpha_j|)-|\alpha_n|\right)+\mu\beta_n\right\}}{|a_n|}$$

Proof of Theorem 5: Consider a polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+\ldots+(a_1-a_0)z+a_0\\&=z^n\Big\{-a_n(z+k-1)+(ka_n-a_{n-1})+(ka_{n-1}-a_{n-2})\frac{1}{z}+\ldots+(ka_{\lambda+1}-a_\lambda)\frac{1}{z^{n-\lambda-1}}\\&\quad+(ka_\lambda-a_{\lambda-1})\frac{1}{z^{n-\lambda}}+(a_1-a_0)\frac{1}{z^{n-1}}+a_0\frac{1}{z^n}-(k-1)\left\{\frac{a_{n-1}}{z}+\frac{a_{n-2}}{z^2}+\ldots+\frac{a_\lambda}{z^{n-\lambda}}\right\}\Big\}\end{aligned}$$

Let $|z|>1$ so that $\frac{1}{|z|^{n-j}}<1$, $j=0,1,2,\ldots$

$$\begin{aligned}|F(z)|\ge|z|^n\{&|a_n||z+k-1|-\{|ka_n-a_{n-1}|+|ka_{n-1}-a_{n-2}|+\ldots+|ka_{\lambda+1}-a_\lambda|\\&+|ka_\lambda-a_{\lambda-1}|+\ldots+|a_1-a_0|+|a_0|+(k-1)(|a_{n-1}|+|a_{n-2}|+\ldots+|a_\lambda|)\}.\end{aligned}$$

Using the fact (for reference see [2]) that for any two complex numbers b_0 and b_1 such that $|b_0|\ge|b_1|$, and $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,2$ and β real

$$|b_0-b_1|\le(|b_0|-|b_1|)\cos\alpha+(|b_0|+|b_1|)\sin\alpha.$$

$$|F(z)| \ge |z|^n \Big[|a_n||z+k-1| - \Big\{k|a_n|\cos\alpha - |a_{n-1}|\cos\alpha + k|a_n|\sin\alpha + |a_{n-1}|\sin\alpha$$
$$+ k|a_{n-1}|\cos\alpha - |a_{n-2}|\cos\alpha + k|a_{n-1}|\sin\alpha + |a_{n-2}|\sin\alpha + ... + k|a_\lambda|\cos\alpha$$
$$-|a_{\lambda-1}|\cos\alpha + k|a_\lambda|\sin\alpha + |a_{\lambda-1}|\sin\alpha + ... + |a_{\lambda-1}|\cos\alpha - |a_{\lambda-2}|\cos\alpha + |a_{\lambda-1}|\sin\alpha$$
$$+|a_{\lambda-2}|\sin\alpha + ... + |a_1|\cos\alpha - |a_0|\cos\alpha + |a_1|\sin\alpha + |a_0|\sin\alpha + ... + |a_0| + (k-1)\sum_{i=\lambda}^{n-1}|a_i|\Big\}\Big]$$

$$= |z|^n \Big[|a_n||z+k-1| - \Big\{k|a_n|(\cos\alpha + \sin\alpha) + (k-1)|a_{n-1}|\cos\alpha + (k+1)|a_{n-1}|\sin\alpha$$
$$+ ... + (k-1)|a_\lambda|\cos\alpha + (k+1)|a_\lambda|\sin\alpha + 2|a_{\lambda-1}|\sin\alpha + 2|a_{\lambda-2}|\sin\alpha + ...$$
$$+ 2|a_1|\sin\alpha - |a_0|\cos\alpha + |a_0|\sin\alpha + |a_0| + (k-1)\sum_{i=\lambda}^{n-1}|a_i|\Big\}\Big]$$
$$= |z|^n \Big[|a_n||z+k-1| - \{k|a_n|(\cos\alpha + \sin\alpha) + (k-1)\left(\cos\alpha \sum_{i=\lambda+1}^{n}|a_{j-1}| + \sum_{i=\lambda}^{n-1}|a_i|\right)$$
$$+ (k+1)\sin\alpha \sum_{i=\lambda+1}^{n}|a_{j-1}| + 2\sin\alpha \sum_{j=\lambda-1}^{1}|a_j| + |a_0|(\sin\alpha - \cos\alpha) + |a_0|\Big\}\Big]$$
$$= |z|^n \Big[|a_n||z+k-1| - \Big\{k|a_n|(\cos\alpha + \sin\alpha) + ((k-1)\cos\alpha + (k+1)\sin\alpha)\sum_{i=\lambda+1}^{n}|a_j|$$
$$+ 2\sin\alpha \sum_{j=\lambda-1}^{1}|a_j| + |a_0|(1 + \sin\alpha - \cos\alpha) + (k-1)\sum_{i=\lambda}^{n-1}|a_i|\Big\}\Big]$$

$$\ge |z|^n \Big\{|a_n||z+k-1| - \Big\{k|a_n|(\cos\alpha + \sin\alpha) + ((k-1)\cos\alpha + (k+1)\sin\alpha)\sum_{i=\lambda+1}^{n}|a_j|$$
$$+ 2\sin\alpha \sum_{j=\lambda-1}^{1}|a_j| + (k-1)\sum_{i=\lambda}^{n-1}|a_i|\Big\}\Big\}$$
$$> 0,$$

if

$$|z+k-1| > k(\cos\alpha + \sin\alpha) + \frac{1}{|a_n|}\{(k-1)\cos\alpha + (k+1)\sin\alpha\}\sum_{i=\lambda+1}^{n}|a_j| + \frac{2\sin\alpha}{|a_n|}\sum_{j=\lambda-1}^{1}|a_j| + (k-1)\sum_{i=\lambda}^{n-1}|a_i|$$

This shows that the zeros of $F(z)$, for which $|z| > 1$ lie in

$$\left|z+k-1\right| \le k(\cos\alpha+\sin\alpha)+\frac{1}{\left|a_n\right|}\{(k-1)\cos\alpha+(k+1)\sin\alpha\}\sum_{i=\lambda+1}^{n}\left|a_j\right|+\frac{2\sin\alpha}{\left|a_n\right|}\sum_{j=\lambda-1}^{1}\left|a_j\right|+(k-1)\sum_{i=\lambda}^{n-1}|a_i|.$$

But those zeros of $F(z)$ for which $\left|z\right| \le 1$ already satisfy the above equation the above equation. We conclude that all the zeros of $F(z)$ and hence of $P(z)$ lie in the disc

$$\left|z+k-1\right| \le k(\cos\alpha+\sin\alpha)+\frac{1}{\left|a_n\right|}\{(k-1)\cos\alpha+(k+1)\sin\alpha\}\sum_{i=\lambda+1}^{n}\left|a_j\right|+\frac{2\sin\alpha}{\left|a_n\right|}\sum_{j=\lambda-1}^{1}\left|a_j\right|+(k-1)\sum_{i=\lambda}^{n-1}|a_i|.$$

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