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Uniformly invariant normed spaces

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Abstract

In this work, we introduce the concepts of compactly invariant and uniformly invariant. Also we define sometimes C-invariant closed subspaces and then prove every m-dimensional normed space with $m > 1$ has a nontrivial sometimes C-invariant closed subspace. Sequentially C-invariant closed subspaces are also introduced. Next, An open problem on the connection between compactly invariant and uniformly invariant normed spaces has been posed. Finally, we prove a theorem on the existence of a positive operator on a strict uniformly invariant Hilbert space.

Keywords: Compactly invariant normed space, Uniformly invariant normed space, Unitary space, Positive operator.

1. Introduction

The subject of extension of linear operators is one of the important subjects in functional analysis. Invariant subspace problem is also. Bandyopadeyay and Roy [1] have studied uniqueness of invariant Hahn-Banach extensions. Author in [2] has studied invariant subspace problem for Banach spaces, In [3, 4] extensions of positive operators has been worked. Saccoman [5] has given a necessary and sufficient condition for extension of a linear operator between Banach spaces. Karapınar [6], has used of Invariants to consider the problem of isomorphic classification of pairs of ℓ-Köthe spaces.

In this work, we introduce the new concepts "compactly invariant and uniformly invariant normed spaces " to prove a theorem on the existence of a positive operator on a strict uniformly invariant Hilbert space. First of all, We have the following definitions and results.

Definition 1.1 [7]: Let X and Y be normed spaces and $T: X \rightarrow Y$ a linear operator. T is called to be a compact operator if $\overline{T(M)}$ is compact, for every bounded subset M of X.

Lemma 1.1 [7]: Let X be a normed space. If $dimX =$, the identity operator I_X is not compact.

Definition 1.2: Let T be a linear operator on a vector space X. If there is a subspace Y of X such that $T(Y) \subseteq Y$ then Y is called an invariant subspace of T.

Throughout this paper, $B(X)$ denotes the normed space of all bounded linear operators on a normed space X. Further, by $K(X)$ we mean that the normed space of all compact operators on X. Clearly, $K(X)$ is a closed subspace of $B(X)$.

Definition 1.3 [5]: A normed linear space X is an unitary space if the norm satisfies the parallelogram low, that is,

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2) \quad \forall x,y \in X$$

The following theorem gives a necessary and sufficient condition for extension of a linear operator between Banach spaces.

Theorem 1.1 [5]: Let X be a Banach space and M be a closed subspace of the real Banach space X and b is a bounded linear operator which maps M into an arbitrary Banach space Y. Then there exist a bounded linear operator B which maps X into Y and $\|b\| = \|B\|$ if and only if X is a unitary space.

2. Main Results

In this section, We let always X be a normed space over $F(\mathbb{R}$ or $\mathbb{C})$; unless the contrary is specified. We Set

$$\mathrm{A} = \{\mathrm{Y} \subseteq \mathrm{X} : \mathrm{Y} \text{ is a closed subspace of } \mathrm{X}\}$$
$$\mathrm{C} = \{\mathrm{T} \in \mathrm{B}(\mathrm{X}) : \mathrm{T}(\mathrm{Y}) \subseteq \mathrm{Y} \text{ for some } \mathrm{Y} \in \mathrm{A}\}$$

and

$$MI(X) = \{\lambda I_X : \lambda \in F\}$$

Definition 2.1: We say that X is compactly invariant when for each $Y \in A$ there exists nonzero $T \in K(X)$ such that $T(Y) \subseteq Y$.

Dealing with the previous definition we have the following theorem.

Theorem 2.1: Let X be a finite-dimensional normed space. Then X is compactly invariant.

Proof. Let $Y \subseteq X$ be an arbitrary closed subspace of X. It is easy to show that the identity operator on X, say $\mathrm{I_X}$, is compact. This completes the proof.

Example 2.1: $\mathbb{R}^n$ and $\mathbb{C}^n$ are compactly invariant normed spaces.

Theorem 2.2: Every infinite-dimensional Banach space contains infinite many compactly invariant subspaces.

Proof. It immediately follows from the Dvoretzky's theorem.[8,theorem8].

Definition 2.2: We say that a closed subspace Y of X is sometimes C-invariant, when there is $T \in C\backslash MI(X)$ such that $T(Y) \subseteq Y$. T is called fixing operator of Y.

Example 2.2: Let $T: X \to X$ be a compact operator. Then $KerT$ is a sometimes C-invariant closed subspace of X.

The problem of the existence of invariant subspaces of a normed space is attractive for many authors, for example see [9-12]. Next, we prove a theorem, in the sense of definition 2.2.

Theorem 2.3: If X is a m-dimensional normed space with $m > 1$, then it has a nontrivial sometimes C-invariant closed subspace.

Proof. Suppose $dimX = n < \infty$. Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis for X. Set $Y = span\{v_1, v_2, \dots, v_{n-1}\}$. Obviously, Y is a nontrivial closed subspace of X. The rest of what we need follows from theorem2.1.

Definition 2.3: We say that a closed subspace Y of a X is sequentially C-invariant when there is a sequence $\{T_n\}$ of $C\backslash MI(X)$ such that $T_n(Y) \subseteq Y$ for all $n \in N$. The sequence $\{T_n\}$ is called fixing sequence of Y.

The next theorem gives an interesting property on invariance in the sense of definition 2.3.

Theorem 2.4: Let Y be a sequentially C-invariant closed subspace of X. Also, let $\{T_n\}$ be fixing sequence of Y which $T_n \to T$ as $n \to \infty$. Then Y is an invariant subspace of T.

Proof. Suppose that $y \in Y$. Since $T_n(Y) \subseteq Y$ for all $n \in N$, so $\{T_n y\}_{n\in \mathbb{N}} \subseteq Y$. By assumption, $lim_{n\to\infty} T_n y = Ty$. On the other hand, $Ty \in Y$. Since $y \in Y$ was arbitrary, so $T(Y) \subseteq Y$.□

The next definition has a key role in the main theorem.

Definition 2.4: We say that a normed space *X* is uniformly invariant when there is an operator $T \in B(X)\backslash MI(X)$ such that $T(Y) \subseteq Y$ for each $Y \in A$. In particular, If *X* is a Hilbert space then we say that it is strict uniformly invariant when furthermore the last assumptions, $T - S$ is positive, for every $S \in B(X)$; then, T is called uniformly invariant operator and strict uniformly invariant operator, respectively.

Open Problem 2.1: Find a normed space which is both uniformly invariant and compactly invariant.

Now we can now prove the main theorem of this paper.

Theorem 2.5: Let *H* be a strict uniformly invariant Hilbert space with strict uniformly invariant operator *T*. suppose that *S* be an linear bounded operator on a closed subspace *Y* of H such that $S(Y) \subseteq Y$. Then there exists a positive operator on *H* such that *Y* is invariant under it.

Proof. By theorem 1.1, there exists a linear bounded operator $\hat{S}$ which maps *H* into *H*. Evidently, the restriction of $\hat{S}$ to *Y* is *S*. Since *H* is strict uniformly invariant, therefore $T - \hat{S} \geq 0$ on H. Set $\tilde{T} = T - \hat{S}$ .Then *Y* is an invariant subspace of $\tilde{T}$, as desired.

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