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From linear to non-linear/chaotic pendulum: a computational study

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Abstract
In this work, we have used computational techniques to examine how the dynamics of a simple pendulum change from linear to non-linear and chaotic. The graph of phase space, angular displacement versus time, and angular velocity versus time are thoroughly examined in our analysis. A significant shift is observed in these representations, particularly in the graph of angular displacement versus time and angular velocity versus time. As the non-linearity is enhanced, we observe a progressive movement from circular to oval shapes in phase space. In damped and forced pendulum scenarios, similar patterns are observed. In these cases, the graphs display a sinusoidal pattern with a diminishing amplitude with time. Surprisingly, in the phase space of the damped pendulum, a spiral type of graph is observed, demonstrating the intricate relationship between damping effects and non-linearity. This research emphasizes the separatrix's function as a crucial cutoff point where the pendulum's motion changes from linear to chaotic.

Keywords
Phase space; Separatrix; Damped pendulum; Trajectory; Runge-Kutta method; Harmonic oscillator

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1 Introduction

Galileo was the first to observe that the swinging time of a lamp in a cathedral remained consistent regardless of its swing's size, at least for the small swings he could observe [1]. Huygens invented the first pendulum clock in 1656 [2], which was a big step forward in timekeeping from earlier techniques.

The pendulum was thus the first oscillator of significant practical significance. The study of pendulum dynamics plays an important role in classical mechanics, and simple harmonic motion (SHM) is well understood for small oscillations. If the particle in the case of periodic motion moves back and forth, then the motion is called oscillatory or vibratory motion. The harmonic motion of the simplest type, having constant amplitude and single frequency, is called simple harmonic motion. The body is said to be in simple harmonic motion (SHM) if the acceleration of the body is directly proportional to the displacement and is directed always toward the mean position. If 'a' be the acceleration of the body and 'x' be the displacement of the body from the mean position, then

$$acceleration \propto displacement$$

i.e.,

$$a \propto x \Rightarrow a = -kx$$

where 'k' is a constant, called a force constant or spring constant, and negative sign indicates acceleration is always directed opposite to the displacement or motion of an object [3].

Suppose a particle of mass 'm' executing SHM. If 'x' be the displacement of the particle from the equilibrium position at any instant of time 't', then from Hooke's law

$$Restoring force(F) \propto x$$

$$\Rightarrow F = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{kx}{m} = -w^2x$$

Where $w^2 = \frac{k}{m}$ The general solution of this equation is

$$x = A_1e^{iw_0t} + A_2e^{-iw_0t}$$

Phase space is the momentum, and position coordinates are set to define the dynamic system. The idea that unifies quantum and classical mechanics is crucial to understanding physics. The space of all possible states for a physical system is known as phase space in classical mechanics. Friedrich Henri Poincare, Ludwig Boltzmann, and Josiah Willard Gibbs invented the idea of phase space in the late 1800 s [4]. A dynamical system's state varies with time, as represented by a point in phase space, and this is known as the phase trajectory. A phase plane that shows the angle θ and the angular velocity $\frac{d\theta}{dt}$ provides a clear visual representation of the dynamics of a pendulum. Certain curves or points, referred to as "attractors", are important in this graphical representation. No matter where the motion begins, attractors are crucial because they symbolize the final states that all potential pendulum trajectories can converge toward over time.

In the case of real-world systems, when large angular displacement and the damping term are taken into account, a simple pendulum exhibits non-linear and chaotic behavior. While many studies have analyzed these behaviors separately, a comprehensive computational study that systematically examines the transition from linear motion to non-linearity and chaos is still lacking. This work has employed the numerical technique Runge-Kutta method to investigate chaotic behavior in pendulums, but further exploration is required to fill the gap between classical and modern computational approaches. This study aims to address this gap by employing a computational framework to analyze phase-space trajectories and quantify the transition from periodic to chaotic motion. Understanding these transitions has important applications in fields such as seismology, robotics, and precision timekeeping, where chaotic oscillations impact system stability and performance.

2 Materials and Methods

A simple pendulum is a heavy-point object suspended by a flexible, weightless, and inextensible string that can oscillate freely in a vertical plane. When a bob of a simple pendulum is displaced through a small angle from its mean position, then it begins to vibrate around its mean position. Such motion of the point mass is called simple harmonic motion.

Suppose a bob of a simple pendulum of mass 'm' whose effective length is 'l'. Let the bob be displaced through 'x' such that the angle θ is very small. Then various forces acting on the bob are:

1. weight 'mg' acting vertically downward.
2. Tension 'T' acting on a string towards the point of suspension

Here, weight 'mg' can be resolved into two components. The component $mgcos\theta$ balances tension T, and another component $mgsin\theta$ is restoring force, which causes the oscillation of the simple pendulum. According to Newton's second law,

$$F = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -mgsin\theta$$

The negative sign indicates that restoring force and displacement are in opposite directions. If we have $x = l\theta$ then above equation becomes

$$ml\frac{d^2\theta}{dt^2} = -mgsin\theta$$

$$\frac{d^2\theta}{dt^2} = \frac{-g}{L}sin\theta \tag{1}$$

Using Maclaurin's theorem, we get,

$$\frac{d^2\theta}{dt^2} = \frac{-g}{L}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ......) \quad (2)$$

Taking the first term of the right-hand side, we get,

$$\ddot{\theta} = \frac{-g}{L}\theta \quad (3)$$

Which has the solution of the form

$$\theta(t) = \theta_0 cos\sqrt{\frac{g}{L}}t \quad (4)$$

Its time period is $T = 2\pi\sqrt{\frac{L}{g}}$ [6]

2.1 Damped harmonic oscillator

A harmonic oscillator is called a damped harmonic oscillator in which the oscillations are damped on account of a resistive or damping force, with its amplitude progressively decreasing to zero. When a simple harmonic oscillator is subjected to a damping force that is proportional to its velocity and an external periodic force, then we write an equation of its motion [7].

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = a_0 sinwt \quad (5)$$

Where $F(t) = ma_0 sinwt$ is the driving force with angular frequency ω.In the absence of driving force i.e., at a = 0, the real solutions of equation (7)at $t \to \infty$ are:

$$x_0(t) = c_1 e^{-(\gamma+\sqrt{\gamma^2-4\omega_0^2})t/2} + c_2 e^{-(\gamma-\sqrt{\gamma^2-4\omega_0^2})t/2}, \quad \gamma^2 - 4w_0^2 > 0 \quad (6)$$

$$x_0(t) = c_1 e^{-\gamma t/2} + c_2 e^{-\gamma t/2}, \quad \gamma^2 - 4w_0^2 = 0 \quad (7)$$

$$x_0(t) = c_1 e^{-\gamma t/2} cos\left(\sqrt{-\gamma^2 + 4\omega_0^2}t/2\right) + c_2 e^{-\gamma t/2} cos\left(\sqrt{-\gamma^2 + 4\omega_0^2}t/2\right) \quad \gamma^2 - 4w_0^2 < 0 \quad (8)$$

If $a_0 > 0$, then the general solution is obtained from the sum of the special solution, $x_s(t)$, and the solution of homogeneous equations is $x_0(t)$. The special solution is obtained as

$$x_s(t) = Asinwt + Bcoswt1001[8]$$

Putting this equation in equation (7), we get,

$$x_x(t) = \frac{a_0[(w_0^2 - w^2)coswt + \gamma wsinwt]}{(w_0^2 - w^2)^2 + \gamma^2 w^2} \quad (9)$$

So, the general solution is given by

$$x(t) = x_0(t) + x_s(t) \quad (10)$$

In the case of resonance without damping $w = w_0$ and $\gamma = 0$. In this case, the solution becomes

$$x(t) = c_1 coswt + c_2 sinwt + \frac{a_0}{4w^2}(coswt + 2wtsinwt) \quad (11)$$

2.2 Forced damped pendulum

Let us consider a simple pendulum in a uniform gravitational field in which its motion is damped by the force that is proportional to its velocity and is under the action of the vertical harmonic external driving force then its equation of motion becomes [9].

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + w_0^2 sin\theta = -2Acoswtsin\theta \quad (12)$$

Where θ is the angle made by the pendulum with the vertical, γ is the damping coefficient, $w_0^2 = g/l$ is the natural angular frequency of the simple pendulum, ω is the angular frequency of driving force. The 2A is the amplitude of the oscillation.

3 Methodology

The current study is both computational and theoretical in nature. It mainly focuses on the phase-space trajectory of the chaotic pendulum. Using the Runge-Kutta method, a numerical solution is obtained. Fortran 90 is used to run the file, and Gnuplot is used to plot the phase space. Fortran is used to program scientific and mathematical applications. The name FORTRAN is an acronym for Formula Translation. It is a high-level programming language. There have been different versions of Fortran since the 1950s when work on it began at IBM. The equation (1) can be solved by using the Runge-Kutta (RK) method [10, 11].From the pedagogical perspective, Fortran90 offers a basic method for numerical computing. It is a well-known language for high-performance numerical computation that is very effective at solving differential equations involving large data sets. Fortran is based on a numerical algorithm that aids researchers and students in expanding their understanding of numerical analysis, whereas Python and MATLAB are based on built-in libraries. Fortran 90 is an excellent tool for solving complicated physical models, such as chaotic pendulums, because it offers a better-optimized solution for iterative numerical computations. The simulation parameters employed in this investigation are the pendulum's length of 2.0 meters, damping coefficient of 0.5, simulation time step of 30 seconds, and angular velocity of 3.1 rad/sec. Even though there are many numerical techniques, such as Euler's method and Verlet integration, we have chosen to employ the Runge-Kutta method (RK) because it offers a higher-order accurate solution and is frequently applied to non-linear dynamical systems because of its enhanced stability. The first-order Euler's approach makes it less precise, requiring a very small time step to minimize errors. It is not dependable for long-term simulations due to its significant truncation errors. Likewise, Verlet integration works well in molecular dynamics simulations but is less appropriate for chaotic differential equations.

4 Results and Discussions

In this study entitled "From Linear to Non-linear/Chaotic Pendulum: A Computational Study," we investigated the changes in a pendulum system's dynamic behavior. Considering the equation $\ddot{\theta} = \frac{-g}{L} sin\theta$ in the beginning, we tracked the pendulum's angular displacement θ against time, which is shown in Figure 2 for the angle of oscillation 90.In the linear regime—where the angular displacement remains small—the pendulum displayed simple harmonic motion, which is characterized by smooth, sinusoidal oscillations. As the amplitude increases, i.e., the angle of oscillation is increased and moved away from basic harmonic behavior, the motion becomes more complex and enters into the non-linear zone. Chaotic dynamics began as the angle grew closer to and beyond 180, at which point the non-linear behavior became more noticeable.

We then looked at the relationship between angular velocity and time, which is shown in Figure 3. A constant amplitude was maintained by the harmonic oscillation of the angular velocity in the linear regime. Significant amplitude and frequency changes were observed as the system entered the non-linear region. Again considering the equation $\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + w_0^2 sin\theta = -2Acoswtsin\theta$, we have plotted the graph of θ versus time, $\dot{\theta}$ versus time and $\dot{\theta}$ versus θ. When the damping term was present, additional complexity was created because energy dissipation changed the oscillatory patterns, which, over time, caused the motion to get slower gradually. Plots showing the relationship between $\ddot{\theta}$ versus θ the phase space analysis, shown in Figure 4, helps us to understand the behavior of the system over time.

For small oscillations, the phase-space trajectory forms closed loops, confirming agreement with theoretical expectations of simple harmonic motion. As the amplitude increases, the deviation from the simple harmonic motion is significant, leading to distortion in the phase space indicating the chaotic behavior. The separatrix is the clear indicator for such transition, marking the boundary between regular and chaotic motion and observed in the phase space plots beyond 180^0. These findings are also in agreement with the theoretical models. When the pendulum enters the chaotic regime, the phase-space diagram exhibits irregular, scattered trajectories, which is consistent with known chaotic behavior in dynamical systems. Similar trends were observed in the case of the damped pendulum, where damping affected the rate of energy dissipation in the system shown in Figure 5, and Figure 6. Plots of the damped pendulum's phase space revealed spiral paths that eventually converged to fixed points, signifying the end of the motion of the system as depicted in Figure 7.

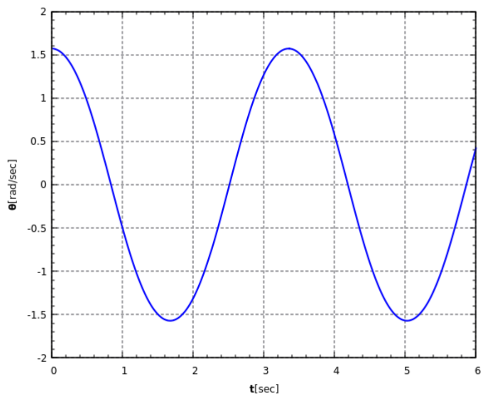


Figure 2: Graph of θ versus t for $\theta = 90$ and dt=0.0

Similarly, the graph of angular velocity $\dot{\theta}$ versuss time is shown in Figure 3.

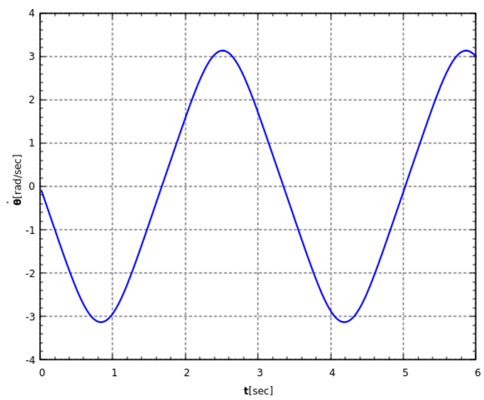


Figure 3: Graph of $\dot{\theta}$ versus t for $\theta = 90$ and dt=0.0

The phase space of the nonlinear pendulum is shown in Figure 4.

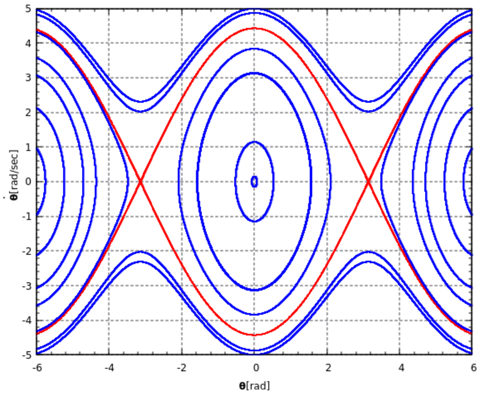


Figure 4: Graph of $\dot{\theta}$ versus t for dt =0.02sec

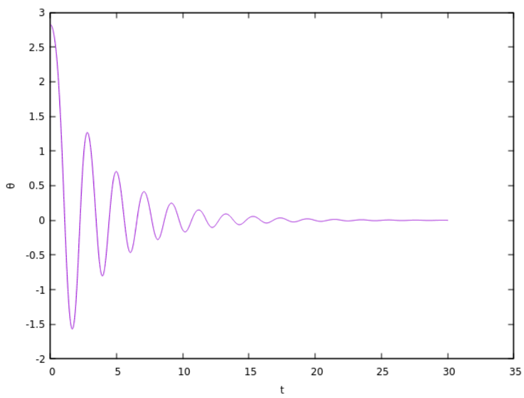


Figure 5: Graph of θ versus t for $\gamma = 0.5$, dt=0.01,$\omega = 3.1$, $\omega_0 = 3.1$

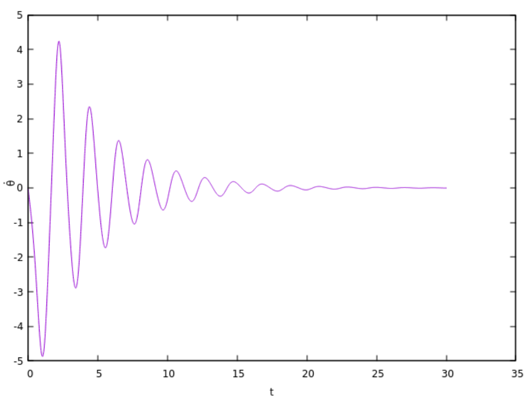
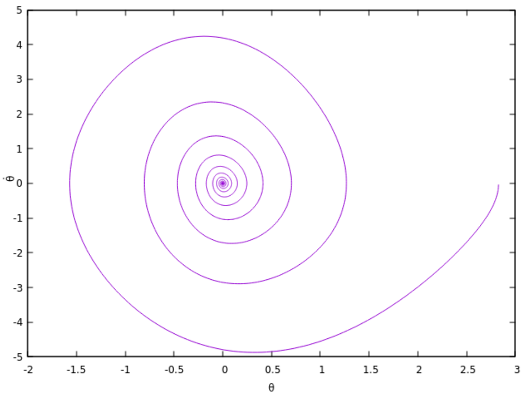


Figure 6: Graph of $\dot{\theta}$ versus t for $\gamma = 0.5$, dt=0.01,$\omega = 3.1$, $\omega_0 = 3.1$



for $\gamma = 0.5$, dt=0.01, $\omega = 3.1$, $\omega_0 = 3.1$

Figure 7: Graph of $\dot{\theta}$ versus θ

5 Conclusions

This study offers insightful information about how linear to non-linear behavior changes in pendulum systems. It clarifies the complex dynamics underlying pendulum motion and advances our understanding of chaotic behavior by fusing rigorous analysis with numerical simulations. The findings have implications beyond theoretical issues and may be used in engineering, physics, and other disciplines. The study of non-linear dynamics and its applications to complex systems requires further investigation.

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