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Richards equation for the assessment of landslide hazards

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Abstract

In this study, we investigate numerical simulation models for water flow in variably saturated (unsaturated) soils. These models are crucial for addressing soil-related challenges and analyzing water-related risks, particularly in the context of water resource management, soil water-induced disasters, and the agricultural impacts of global environmental changes. The Richards equation is one of the most widely used models for simulating water flow in porous media, especially in unsaturated soils. However, as a highly nonlinear parabolic partial differential equation (PDE), it has limited analytic solutions, which often lack precision in practical scenarios. This necessitates the development of innovative and robust numerical methods for accurate simulations. We introduce a numerical procedure that linearizes the Richards equation using the Kirchhoff integral transformation, followed by discretization with various time-stepping schemes. This approach enables efficient and accurate modeling of water flow. To extend its application, we integrate the numerical solution with a hydrological infinite slope stability model to evaluate landslide hazards. Specifically, we calculate the factor of safety index based on an axisymmetric form of the Richards equation, which helps identify potential landslidenprone areas. Furthermore, our model provides a framework for predicting landslides by considering the interplay between water flow and the physical, geological, and topographical characteristics of a landscape. This integrated approach offers valuable insights for geohazard assessment and the mitigation of water-induced risks.

Keywords

Richards equation; Moisture content; infinite sloe model; Landslide hazards; Factor of safety. .

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1 Introduction

Landslides occur in many regions worldwide, particularly in mountainous areas. Geologically, the Himalayan mountain chain, stretching approximately 2,400 kilometers, is among the most tectonically active mountain ranges on Earth [1]. This vast terrain is home to millions of people and diverse ecosystems, spanning countries such as Nepal, Bhutan, northern Pakistan, northern India, and other parts of Asia. The Himalayan region faces significant land sliding problems due to a combination of factors: uneven topography, dynamic geological structures, fragile and fractured rock formations, and heavy, concentrated rainfall during the monsoon season. These elements collectively contribute to frequent and severe landslides, posing substantial risks to lives, infrastructure, and the environment in this region. The increasing frequency of severe and concentrated rainfall events, largely driven by climate change, has significantly exacerbated landslide risks worldwide, including in the Himalayan region. Heavy rainfall, a primary trigger for infiltration-induced landslides, is a common occurrence across this mountainous terrain. These landslides lead to extensive destruction of natural infrastructure, damage to roads, houses, and other man-made constructions, and result in substantial annual economic losses and fatalities. In unsaturated soils, infiltration-induced landslides typically occur close to the ground surface, where heavy rainfall triggers shallow landslides. These shallow landslides commonly develop within the vadose zone, the unsaturated region above the water table, often destabilizing surface layers in foothill areas. Slope failure, which governs the stability and instability of soil surfaces, is closely linked to the infiltration process. Fluctuations in matric potential within unsaturated soils play a crucial role in this process. As rainfall infiltrates the soil, the resulting changes in matric potential reduce mechanical stability, triggering landslides. Understanding these dynamics is critical for developing strategies to mitigate landslide hazards in regions prone to heavy rainfall. Infiltration-induced landslides in the Himalayas cause severe damage to lives, property, infrastructure, and the environment, especially during the monsoon season. To address these challenges, it is essential to examine the factors, underlying issues, and environmental impacts associated with such landslides. The primary cause of these landslides is surface failure, which is influenced by the interconnectedness of slope instabilities. Analyzing slope stability in unsaturated soils is crucial, as it provides insights into the mechanical behavior of slope materials. A detailed study of these materials enables the calculation of the factor of safety (S_F), a conventional measure for assessing the stability of sliding surfaces along failure planes. Certain weather conditions, including continuous and heavy rainfall, rapid snow melt, and abrupt weather changes, are key drivers of slope instability. These phenomena are closely tied to fluctuations in pore water pressure within the soil. During infiltration and redistribution processes, such as rainfall and post rainfall periods, pore water pressure exhibits abnormal variations. Increased pore saturation leads to a rise in pore water pressure, which, when repeated cyclically, reduces the soil's effective stress and triggers slope failure. The Richards equation, a nonlinear partial differential equation, effectively governs these processes and can be applied for quantitative landslide hazard assessment. By coupling the Richards equation with a slope stability model, it is possible to develop a robust framework for analyzing slope failure in relation to weather data. This integrated approach provides a powerful tool for capturing the dynamics of slope stability under various environmental conditions [2]. The accuracy and reliability of this method depend heavily on the quality and availability of soil parameters, weather data, and the physical characteristics of the landscape. By incorporating these factors, a dependable procedure for landslide hazard assessment can be tailored to specific regions, enabling precise and effective mitigation strategies [3].

In this work, we propose an approach to modeling infiltration induced landslides, focusing on the critical role of infiltration processes in predicting landslide hazards. To achieve this, the Kirchhoff transformed Richards equation [4] is integrated with the infinite slope stability model [5], providing a robust framework for landslide prediction. This approach incorporates various hydraulic conductivity functions, which are experimentally verified and describe the movement of water through pore spaces. These functions depend on soil and fluid properties, such as intrinsic permeability, density, viscosity, degree of saturation, shear stress, and strength. The relationships between hydraulic conductivity and moisture content are captured using established empirical formulations [6], which are critical for understanding soil behavior under different conditions. To account for slope stability, a hydrological slope stability model is coupled with the nonlinear Richards equation. This integration enables the development of a comprehensive formulation for assessing landslide hazards. The new formulation emphasizes infiltration-induced landslides in unsaturated soils and captures the complex interplay between soil moisture dynamics and slope stability. The validity of the proposed approach will be tested by comparing its predictions with interpolated exact data. Applications and examples will also be conducted to illustrate the practical utility of the method, demonstrating its effectiveness in accurately predicting landslide hazards in diverse scenarios.

2 Background

This work aims to provide an integrated understanding of hydrogeology and the mathematical framework necessary for studying water flow in unsaturated soils and infiltration- induced landslide hazards. A key focus is on deriving water flow relations for unsaturated soils from fundamental equations at the pore scale, offering a foundation for developing governing equations for modeling water flow [4]. The study of water flow in unsaturated porous media requires a well-defined formulation of governing equations, supported by appropriate constitutive relationships [6], and accompanied by specific boundary and initial conditions to complete the model. Typically, water flow through unsaturated soil is described using the Richards equation, which is based on semi-empirical foundations first developed by Buckingham (1907) [7] and later refined by Richards (1931) [4]. Despite its limitations and drawbacks, the Richards equation remains the most widely used model for simulating water flow through both saturated and unsaturated soils. Its long nstanding history of application highlights its significance, particularly in infiltration studies across various disciplines. However, solving the Richards equation is challenging due to the nonlinearities in constitutive relationships, especially those involving hydraulic properties. While analytical solutions exist, they are rare and limited to specific cases requiring simplifications or substitutions. Consequently, numerical solutions have become the dominant approach, with numerous techniques developed to address these challenges. However, achieving high accuracy, robustness, and cost effectiveness remains an ongoing effort. This work emphasizes the critical role of solving the Richards equation in understanding complex flow phenomena in unsaturated soils. The discussion extends to the numerical challenges and computational advancements required for robust solutions. To address these issues, a qualitative analysis is presented to better understand the behavior of the Richards equation and its implications for modeling and simulation. Lots of numerical solution techniques were employed efficiently to solve the Richards equation with the use of Newton–Raphson or Picard iterative techniques together with finite difference or finite element or finite volume approximations. Finite difference methods were used by Celia and Bouloutas [8] or Woodward and Dawson [9] or Dogan and Motz [10]. Finite element methods are found in Celia and Bouloutas [8], Lehmann and Ackerer [[11], Simunek [12], Forsyth et al. [13], Katvetsi et al. [14]. Mixed finite elements were used by Diersch and Perrochet [15], Knabner and Schneid [16], Chavent and Roberts [17]or Fahs et al. [18]. Manzini and Ferraris [19], Orgogozo et al. [20] or Caviedes–Voulli'eme et al. [21] or C.E. Zambra et al. [22] were used finite volume method. Primary variable switching between the pressure head and water content; upstream weighting of the relative permeability; methods of lines and transformation techniques were used by Celia et al. [8]; Kirkland et al. [23]; Tocci et al. [24]; Williams et al. [25]; Mathews et al. [26]; Ji et al. [27]; Li et al. [28]; Zha et al. [29]; Zhang et al. [30]. Generally, numerical methods are adopted to the pressure head–based form of Richards equation in saturated, unsaturated and for layered soils. However, there is a poor mass balance for unsaturated soils with unacceptable time stepping limitations. The methods which are used in pressure head–based form are also applied to mixed form of Richards equation by Celia et al. [8]with a strategy to evaluate the change in water content over one time step directly from the change of the pressure head. It was found that in this strategy, there is a good mass balance. But according to Kirkland et al. [23] under very dry initial conditions, there was a CPU efficiency problem in this method. Numerical methods of the moisture–based form of Richards equation has a significantly improved performance when the infiltration phenomenon into very dry soil (Hills et al. (1989) [31], Kirkland et al. (1992) [23]; Forsyth et al. (1995) [13]; Zha et al. (2013) [32]). But the topmost drawback of this method is that it can be used for simulating water flow in unsaturated region only. To capture and retain the beneficiary of the moisture based method Wu and Forsyth (2001) [13]; Kirkland et al. [23] and Zeng et al. (2018) [33] were improved by combining these two. Kirkland (1992) [23] was introduced a new variable having different characteristic at unsaturated and saturated region in which it may be a linear function of moisture content for unsaturated soil and a linear function of pressure head for saturated or nearly saturated soil. For relatively dry initial conditions this method has a very good CPU efficiency found by Williams et al. (2000) [25]. Moreover, there should be defined a point in which these two methods are joined which may directly affect the accuracy and robustness of the numerical results.

Haverkamp et al. (1977) [6]; Campbell (1985) [34]; Vauclin et al. (1979) [35]; Ross and Bristow (1990) [36]; Williams et al. (2000) [25]; Ji et al. (2008) [27]; Berninger et al(2011) [37]; Zhang et al. (2015, 2016, 2018) [30]; Li et al. (2016) [28] used Kirchhoff integral transformation approach to solve the non-linear Richards equation. Pan and Wierenga (1995) [38] laid out a special numerical method to overcome the difficulties of evaluating the integral in Kirchhoff transformed approach. A second integral transform formula was introduced by Williams et al. (2000) [25], interpolating the linear combination of moisture content with an arbitrary constant and the Kirchhoff integral transform. Indeed, this method had really a good efficiency and robustness as compared to others. To solve unsaturated flow equation with Kirchhoff transformation Zhang et al. (2015) [30] introduced a new numerical method as a good name Finite Analytic Method (FAM). In this method, a local analytic solution is used to form a set of algebraic equation on the Kirchhoff transformed variable. This method established an accurate and efficient solution. Zhang et al. (2016) [30], applied FAM on mixed form of Richards equation. Also Zhang et al. (2018) [30], extended it into the two–dimensional Richards equation. However, its good achievement, the Finite Analytic Method is limited only for homogeneous porous media soil. Ji et al. (2008) [27], used the transient pressure head–based method and compared a steady state solution obtained from Kirchhoff transformation to a steady–state solution obtained from time marching scheme. They used Gardner's constitutive relationship [39] and their result showed that the computational cost in unsaturated flow simulation can greatly be reduced and the use of Kirchhoff transformation has great implication for subsurface water problems. But they only considered pseudo–heterogeneous layered porous media and limited it only to steady–state conditions. Although, Kirchhoff transformation method has some limitations, it seems to be very promising method for simulation of Richards equation in unsaturated porous media region. Following into this method, numerical errors will be reduced effectively because variations of Kirchhoff head in its integral nature are much smaller than those in pressure head. Indeed, when the Kirchhoff transformation is employed along with some specific constitutive relations to solve nonlinear Richards equation, a full linearization of the unsaturated Richards equation is permitted. This linearization has an ability of analytic and semi–analytic solutions. Furthermore, Numerical Methods for solving nonlinear Richards equation are pertaining to specific constitutive relationships. The popular widespread constitutive relationships were developed by Gardener, van Genuchten, Brooks and Corey and Haverkamp et al. [6]. Correspondingly, they gave the well measured relationship between hydraulic conductivity, pressure and moisture content. For example, Heejun Suk et al. (2019) [40], obtained a numerical solution of the Kirchhoff transformed Richards equation in variably saturated flow in heterogeneous layered soil. This method has followed the Gardener's relationships and is used to solve the linearize partial differential equation. Egidi et al. (2018) [41], used van Genuchten consecutive relationship for a numerical solution of Richards equation providing a simple method adaptable in parallel computing. A constitutive relationship developed experimentally by Haverkamp et al. were used by Liu Fengnan et al. (2020) [42]to a linearized finite difference scheme for the Richards equation under variable–flux boundary conditions. Among these, a most robust numerical method an explicit stabilized Runge–Kutta–Legendre Super Time–Steeping scheme is developed [42]to solve mixed form of nonlinear Richards equation with no source and sink terms and with sink terms as evapotranspiration [43] numerically adopting the constitutive relationship developed by Haverkamp et al. [6]. The scheme is based on the natural phenomena taking into account the hydraulic conductivity as a function of water pressure head as a real situation with experimental data. Finally, Landslides in the Himalayan region are driven by a variety of interconnected processes and causes. The primary factors contributing to landslide hazards include geological, morphological, physical, and human-induced causes. Among these, the foothills of the Himalayas are particularly vulnerable to landslides triggered by unusual, heavy, and concentrated precipitation during the monsoon season. Rainfall is the most significant triggering factor, with the onset of the monsoon frequently leading to widespread landslide occurrences and disasters. These events pose severe risks to people, infrastructure, and ecosystems, affecting countless lives and livelihoods on both large and small scales throughout the region. This paper is organized as follows : in section 3, we present the methodology along with Kirchhoff transformed axi-symmetrical form of Richards equation, including the numerical methods based on finite difference schemes with different time - stepping and the slope failure models for landslide stability in unsaturated soils. In section 4, we show the results of the numerical procedure implemented in python. In section 5, we discuss the results and the future development of this work.

3 Methodology

The flow of water in unsaturated soil follows the Darcy law and mass conservation equation [4]. Thus, the flow phenomena can be expressed by Richards equation, a highly non linear PDE. However, the analytical solution of this equation is impossible, we established a robust numerical solution of this equation in (1+1)D and (2+1)D by adopting different time - stepping schemes and compared their results and used the appropriate one to find the factor of safety for the prediction landslide in the potential landslide area. We have used the (1+1)D solution for bench marking and obtained the solution for axi-symmetrical form of this equation.

3.1 Richards equation

The axi-symmetrical form of Richards equation can be written as

$$\frac{\partial \theta}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rK(\psi)\frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(K(\psi)\frac{\partial \psi}{\partial z}\right) + W, \tag{1}$$

where $\theta(\psi)$ and $K(\psi)$ represent the moisture content $\theta(\psi)$ and hydraulic conductivity $K(\psi)$, respectively, and W is the rate of precipitation.

The function $\theta(\psi)$ and $K(\psi)$ are taken from empirical formulas, the Van Genuchten and Haverkamp et al. models [6, 44, 45] are the most widely used in scientific computations due to their formulation by smooth function. Commonly used versions of these models can be written as:

$$K(\psi) = K_s \left(1 - (\alpha\psi)^{n-1}\right)^2 / \left(1 + (\alpha\psi)^n\right)^{m/2}, \tag{2}$$

$$\theta(\psi) = \theta_r + \frac{\theta_s - \theta_r}{(1 + (\alpha\psi)^n)^m}, \tag{3}$$

where α is the reciprocal value of ψ_0, i.e., the air entry point [46], θ_s and θ_r are the saturated water content and residual water content respectively. K_s is the value of the permeability when the soil is saturated. m, n are empirical parameters depending on the soil, also m and n have the relation:

$$m = 1 - \frac{1}{n}. \tag{4}$$

The various hydraulic parameters, e.g., α, θ_r, θ_s, n, K_s, are related to the pore size distribution and pore geometry, so they ultimately depend on the soil type and can be determined experimentally by specific laboratory tests. For implementation and to improve the results, we used the Van Genuchten model.

3.2 Linearization Techniques

Applying the Kirchhoff integral transformation in equation 1, we let $h = \psi - z$ and define:

$$\phi(h) = \int_0^h \overline{K(h')}\mathrm{d}h'. \tag{5}$$

Since $K(h) > 0$ from equation (2), the function $\phi(h)$ is strictly increasing with $\overline{K}(h) = K(\psi)$. Taking derivatives of both sides of the transformation with respect to r and z, we obtain:

$$\frac{\partial\phi}{\partial r} = K(\psi)\frac{\partial\psi}{\partial r}, \quad \frac{\partial\phi}{\partial z} = K(\psi)\frac{\partial\psi}{\partial z}. \tag{6}$$

Again, taking the derivative of equation (6) with respect to r:

$$\frac{\partial^2\phi}{\partial r^2} = K(\psi)\frac{\partial^2\psi}{\partial r^2} + \frac{\partial K}{\partial\psi}\left(\frac{\partial\psi}{\partial r}\right)^2, \tag{7}$$

and

$$\frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial h}\frac{\partial h}{\partial z} = \overline{K}(h)\frac{\partial(\psi - z)}{\partial z} = K(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right) = K(\psi)\frac{\partial\psi}{\partial z} - K(\psi) \tag{8}$$

Again differentiating of equation (8),

$$\frac{\partial^2\varphi}{\partial z^2} = \frac{\partial}{\partial z}\left(K(\psi)\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}(K(\psi)) \tag{9}$$

Using the equations (6),(8) and (9), the Richards equation (1), takes the form

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} + W \tag{10}$$

with $\overline{\theta}(\varphi) = \theta(h)$. The corresponding initial and boundary conditions for the transformed equation (10) takes the following form

$$.\theta(\text{r, z, } 0) = \varphi_0(\text{r, z}), \text{ Rin } \le r \le \text{Rout}, \ 0 \le \text{z} \le \text{Ztop}$$

$$\begin{aligned} \frac{\partial\varphi}{\partial z} &= \overline{q}(t), \text{ r} > 0, \text{ z} = 0, \text{ t} > 0 \\ \varphi(\text{r, Zbot, t}) &= \overline{\beta}(t), \text{ r} > 0, \text{ t} > 0 \\ \frac{\partial\varphi}{\partial r} &= 0, \text{r} = \text{Rout}, \text{ z} > 0, \text{ t} > 0 \\ .\varphi(\text{Rin,Z, t}) &= \overline{\beta_1}(t), \text{ t} > 0 \end{aligned} \tag{11}$$

The Kirchhoff transformation transformed the nonlinear equation (1) to a nonlinear parabolic equation (10). Also we note that the Kirchhoff transformation preserves the uniqueness of the solution for the transformed problem.

3.3 Discretization techniques and implementation

We have the transformed equation is in two different state variables. To solve the transformed equation (10) numerically with the prescribed initial and boundary conditions (11), it is feasible to have a single state variable. For this, and are assumed as single valued continuous functions of one another, and arranging these variables as

$$\frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial\varphi}\frac{\partial\varphi}{\partial t} = \left(\frac{1}{\frac{\partial\varphi}{\partial\theta}}\right)\frac{\partial\varphi}{\partial t}, \ \frac{\partial\varphi}{\partial\theta} = \frac{\partial\varphi}{\partial h}\frac{\partial h}{\partial\theta}, \tag{12}$$

Differentiating (3) with respect to h, we get

$$\frac{\partial\theta}{\partial h} = \alpha^n(\theta_s - \theta_r)(-m)(1 + |\alpha h|^n)^{-m-1}n|h|^{n-1}$$

$$\frac{\partial\varphi}{\partial h} = \overline{K}(h) = K(\psi), \tag{13}$$

Using (12) and (13), the transformed Richards equation (10) takes the form

$$c(\varphi)\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} \tag{14}$$

where the functional coefficient c depends on through h as

$$c(\varphi(h)) = -\frac{\alpha^n \mathrm{mn}(\theta_s - \theta_r)|h|^{n-1}}{\overline{K}(h)(1+|\alpha h|^n)^{m+1}} \quad (15)$$

3.4 Finite Difference Scheme

We set up a two dimensional $(r,\ z)$ uniform grid for an axi symmetric problem in the cylinder geometry by subdividing the radial length $[Rn,\ Rout]$ into Mr subintervals of width $\Delta r = \frac{R_{\mathrm{out}} - R_{\mathrm{in}}}{M_r}$ and the height $[0, Z_{\mathrm{top}}]$ in to $\mathrm{M_z}$ subintervals of width $\Delta z = \frac{\mathrm{Ztop}}{M_z}$. We construct a grid (r_i, z_j, t_n) with $r_i = i\Delta r, i = 0, 1, 2, ..., M_r, z_j = j\Delta z, j = 0, 1, 2, 3, ...M_z$, and $t_n = n\Delta t, n = 1, 2, 3..., N$. Let $\varphi^{n_{i,j}}$ denote $\varphi(r_i, z_j, t_n)$. The partial differential equation (14) can be approximated using forward difference in time and central difference in space as

$$\begin{aligned} \frac{\partial \varphi}{\partial t}|_{(r_i z_j, t_n)} &\approx \frac{\varphi^{n+1_{i,j}} - \varphi_{i,j^n}}{\Delta t}, \\ \frac{\partial \varphi}{\partial r}|_{(r_i z_j, t_n)} &\approx \frac{\varphi^{n_{i+1,j}} - \varphi_{i-1,j^n}}{2\Delta r} \\ \frac{\partial^2 \varphi}{\partial r^2}|_{(r_i z_j, t_n)} &\approx \frac{\varphi^{n_{i-1,j}} - 2\varphi_{i,j^n} + \varphi^{n_{i+1,j}}}{\Delta r^2}, \\ \frac{\partial^2 \varphi}{\partial z^2}|_{(r_i z_j, t_n)} &\approx \frac{\varphi^{n_{i,j-1}} - 2\varphi_{i,j^n} + \varphi^{n_{i,j+1}}}{\Delta z^2} \end{aligned} \quad (16)$$

Using a weighted average of the derivatives $\left(\frac{\partial \varphi}{\partial r}, \frac{\partial^2 \varphi}{\partial r^2}, \frac{\partial^2 \varphi}{\partial z^2}\right)$ at the two time levels, $\mathrm{t_n}$ and $\mathrm{t_{n+1}}$ and adopting Crank-Nicolson(CN) scheme , equation (14) can be discretized as

$$\begin{aligned} \frac{\varphi^{n+1}_{i,j} - \varphi^n_{i,j}}{\Delta t} &= \frac{1}{2c^n_{i,j}(\Delta r)^2}\left[\varphi^{n+1}_{i-1,j} - 2\varphi^{n+1}_{i,j} + \varphi^{n+1}_{i+1,j}\right. \\ &+\varphi^n_{i-1,j} - 2\varphi^n_{i,j} + \varphi^n_{i+1,j} + \frac{1}{2c^n_{i,j}(\Delta z)^2}\left[\varphi^{n+1}_{i,j-1} - 2\varphi^{n+1}_{i,j}\right. \\ &+\varphi^{n+1}_{i,j+1} + \varphi^n_{i,j-1} - 2\varphi^n_{i,j} + \varphi^n_{i,j+1} + \frac{1}{4r_i c^n_{i,j}(\Delta r)}\left[\varphi^{n+1}_{i+1,j}\right. \\ &- \varphi^{n+1}_{i-1,j} + \varphi^n_{i+1,j} + \varphi^n_{i-1,j}\Big] \end{aligned} \quad (17)$$

We collect the unknowns on the left hand side :

$$\begin{aligned} &-\left(F_r - \frac{F}{r_i}\right)\varphi^{n+1}_{i-1,j} + (1+2F_r+2F_z)\varphi^{n+1}_{i,j} - \left(F_r + \frac{F}{r_i}\right) \\ &\qquad \varphi^{n+1}_{i+1,j} - F_z(\varphi^{n+1}_{i,j-1} + \varphi^{n+1}_{i,j+1}) \\ &= \left(F_r - \frac{F}{r_i}\right)\varphi^n_{i-1,j} + (1-2F_r-2F_z)\varphi^n_{i,j} + \left(F_r + \frac{F}{r_i}\right) \\ &\qquad \varphi^n_{i+1,j} - F_z(\varphi^n_{i,j-1} + \varphi^n_{i,j+1}) \end{aligned} \quad (18)$$

where $F_r = \frac{\Delta t}{2c^n_{i,j}(\Delta r)^2}, F_z = \frac{\Delta t}{2c^n_{i,j}(\Delta z)^2}, F = \frac{\Delta t}{4c^n_{i,j}(\Delta r)}$.

The equation (18) is coupled at the new time level $n+1$. That is, we must solve a system of (linear) algebraic equations, which we will write as $Ax = B$, where A is the coefficient matrix, x is the vector of unknowns, B is the right hand-side.

To solve the system of linear equations, we need to form a matrix system $Ax = B$, where the solution vector x must have one index. For this, we need a numbering of the unknowns with one index, not two as used in the mesh. We introduce a mapping *position* $(i,\ j) = v(i,\ j)$ from a mesh point with indices $(i,\ j)$ to the corresponding unknown p in the equation system.

$p = v(i,\ j) = j(Vr + 1) + i$, *for* $i = 0,\ 1,\ 2,\ \cdots,\ Vr,\ j = 0,\ 1,\ 2,\ \cdots,\ Vz,$

With this mapping, we number the points along the radial direction starting with $z = 0$ and then filled one mesh line at a time. In another way

$p = v(i,\ j) = i(Vz + 1) + j$, *for* $i = 0,\ 1,\ 2,\ \cdots,\ Vr,\ j = 0,\ 1,\ 2,\ \cdots,\ Vz.$

with $r = 0$ and then filled one mesh line at a time. From this we can get the general feature of the coefficient matrix obtained from the discretized equation (18).

Let $A_{k,l}$ be the value of element $(k,\ l)$ in the coefficient matrix A, where k and l are the numbering of the unknowns in the equation system. We have $A_{k,l} = 1$ for $k = l$ corresponding to the all known boundary values. Let k be $v(i,\ j)$, i.e., the single index corresponding to the mesh point $(i,\ j)$. Then, for interior mesh along with boundary, we have

$$\begin{aligned} A_{v(i,j),v(i,j)} &= A_{k,k} = 1 + (F_r + F_z), \\ A_{k,v(i-1,j)} &= A_{k,k-1} = -F_r \\ A_{k,v(i+1,j)} &= A_{k,k+1} = -F_r \\ A_{k,v(i,j-1)} &= A_{k,k-(v_z+1)} = -F_z \\ A_{k,v(i,j+1)} &= A_{k,k+(v_z+1)} = -F_z \end{aligned}$$

The corresponding right hand side vector in the equation system has the entries B_k, where k numbers the equations with the given boundary values.

The above mention algorithm can be used to update the transformed variable $\varphi^n_{i,j}$

to its value in the next time level $\varphi^{n+1}_{i,j}$. But we cannot advance the algorithm to the next time level $\varphi^{n+2}_{i,j}$ without evaluating the function $c(\varphi^{n+1}_{i,j})$

which requires computing the intermediate variable $h_{i,j}^{n+1}$. For this , we imply the equation (13) which can be approximated as

$$h_{i,j}^{n+1} = h_{i,j}^{n} + \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{K(h_{i,j}^{n})}. \tag{19}$$

3.5 Landslide Stability analysis in unsaturated soil

This study focuses on the modeling of infiltration-induced landslides in unsaturated soils, specifically analyzing slope stability and instability using the axi-symmetric Kirchhoff-transformed Richards equation. The approach considers the variation of moisture content with water pressure head and vice versa. Infiltration- induced landslides on ground surfaces often resemble shallow landslides triggered by intense rainfall events. The problem is addressed using an infinite slope model to calculate the safety factor ($\mathrm{S_F}$), which serves as a key indicator of slope stability. The safety factor is defined as the ratio of resisting forces that prevent slope failure to driving forces that cause collapse. As ($\mathrm{S_F}$) greater than one indicates stability, while as ($\mathrm{S_F}$) less than one signals instability. This numerical value acts as a landslide hazard index, enabling predictions about the likelihood of landslides for given topographic conditions. The infinite slope model is adopted due to its simplicity and effectiveness, especially for shallow landslides where the failure plane is parallel to the ground surface and relatively shallow compared to the slope length. For inter connected to the moisture profile obtained from the above solution schemes we preferred the following hydrological infinite slope model

$$S_F = \frac{C}{\mathrm{zY}_t \sin\beta\cos\beta} + \frac{\tan\varphi}{\tan\beta} + \frac{h\theta Y_w \tan\varphi}{\mathrm{zY}_t \sin\beta\cos\beta} \tag{20}$$

where C is the effective cohesion, is the moisturity, Yt is the unit weight of the soil, Yw is the unit weight of the water, is the angle of internal friction, z is the soil thickness, is the slope of the inclined surface. The combination of the infinite slope model (Equation 20) with Richards equation, allows the landslide hazard evaluation directly from weather data. In particular, the solution of Richards equation will be used to obtain information about in the expression.

4 Simulation Results

4.1 Numerical Setup

To demonstrate the numerical solution of the prescribed model, the numerical process developed in the above section is written in python and ran on a laptop with 2.8 GHz Quad-Core Intel Core i7 processor. We considered a specific infiltration experiment in unsaturated soil and observe the corresponding outcomes. At first the setup consisted on a landslide area having dimension 70m $\times$ 100m. The physical constituent of the landslide area is sandy soil. We use the soil parameters and characteristics relationship between the soil moisture content () and the hydraulic conductivity $K()$ from the work of Van Genuchten.

$$K(\psi) = \mathrm{Ks}\left[1 - |\partial\psi|^{n}\right]^{-\frac{m}{2}} \left[1 - \left(\frac{||^{n}}{1 + ||^{n}}\right)^{m}\right]^{2} \tag{21}$$

$$\theta(\psi) = \theta_r + \frac{(\theta_s - \theta_r)}{(1 + ||^{n})^{m}} \tag{22}$$

The simulation starts with a uniform saturation $\theta = 0.1\,\mathrm{cm^3/cm^3}$ and a constant water head $\psi = -61.5\,\mathrm{cm}$ is maintained at the bottom boundary $z = Z_{\mathrm{bot}}$. For the upper boundary $z = Z_{\mathrm{top}}$ at the soil surface, a constant flux $q(t) = 13.69\,\mathrm{cm/hr}$ for $t < 0.7\,\mathrm{hr}$ and zero normal flux condition for $t > 0.7\,\mathrm{hr}$ is maintained. To compute the approximate solution using different time-stepping schemes, we used a uniform spatial step size $\Delta z = 2\,\mathrm{cm}$ in the axial direction and a step size of $\Delta z = 2\,\mathrm{cm}$ in the radial direction. The simulation was run for $0.75\,\mathrm{hr}$.

For the infiltration-induced landslide, we interpolated the moisture variation obtained from the above experiment to the hydrological model for stability analysis. Data was obtained from the Department of Hydrology and Meteorology, Government of Nepal, and field observations in the landslide area located in Yangbarak VDC, Ward No. 6, Shripung, Panchthar district. Rainfall data from the region was applied in our simulation model (Equation 20).

Figure 1. Landslide area located at Yangbarak VDC(Photo courtesy: Ganga)

5 Results and Discussion

The axi symmetrical model was employed to simulate vertical wa- ter infiltration into an unsaturated homogeneous porous medium and to analyze the landslide hazards, hydrological model was used. Numerical experiments were carried out by using sand column (proposed by Van Genuchten.)

with finite difference schemes(FTCS, BTCS, CN and RKL). Since we do not have exact solution, we have used numerical solution obtain from above (RKL) finite difference scheme for one dimensional as the reference solution Timsina, R.C et al. [42] for numerical experiment of two dimensional water infiltration. Fig. (2) (a) shows the variation of volumetric moisture content in sand for one dimension, and Fig. 2 (b) shows the comparison of different time-steeping schemes as discussed in [42]. In this numerical experiment, the function used for the hydraulic conductivity and the water content as in one dimensional case were taken from Van Genuchten [46].

We set the axi-symmetrical form of Richards equation in an unsaturated flow into a region of sandy soil. For this, we suppose the domain is specified to be an annulus, $r_{\text{inner}} \leq r \leq r_{\text{outer}}$ with r_{inner} strictly greater than zero, other than a disk. The annular domain consists of 70 cm in length with 100 cm of annular radius. We suppose it vertically downward where the z-axis is taken as downward positive with a constant water head $\psi = -61.5$ cm at the left (Dirichlet) boundary condition $r = R_{\text{in}}$ as dummy. A constant flux boundary condition (Neumann) on the right $r = R_{\text{out}}$ is taken and a constant water head $\psi = -61.5$ cm at the bottom boundary $z = Z_{\text{bottom}}$. At the upper boundary $z = 0$ (the soil surface), a constant flux $q(t) = 13.69$ cm/hr for $t < 0.7$ hr and zero normal flux condition for $t > 0.7$ hr. The solution domain was meshed using a spatial step size of $\Delta z = 2$ cm on the axial direction and a step size of $\Delta r = 2.5$ cm on the radial direction and the simulation was run for 1 hr. The longitudinal water content profiles and radial water content profiles at time 0 hr and some other times are shown in Figure (3) (top). From this figure, it is possible to note that the residual water content at the bottom of the annulus is reached when time t is about 0.50 hr. The moisture content contours are shown in Figure (4) and corresponding surface plots of moisture content are in Figure (4) (left).

We observe the flow phenomena and carried out the variation of water content in the region we have prescribed. We have interconnected the above result to the infinite slope model to evaluate the safety factor. We have measured the slope angle and the thickness of the unsaturated soil as 30^0 and 8m. We use the parameter of effective cohesiveness is 4.4kPa, the frictional angle is 28^0, the soil unit weight is 23.4kN/m^3. Figure 6 depicts the computed results of safety factor for different pressure head observed in the experiment. The factor of safety obtained by the above model demonstrates the possibilities of surface failure and operation of landslide hazards. From figure 6 we conclude that the situation of surface failure may increases when the level of ground water is increased by heavy rainfall meanwhile the pressure head energy increase and the level of unsaturated region decrease and the possibilities of landslide increases.

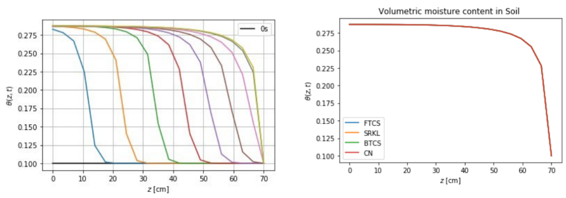

Figure 2: Variational trend of moisture content in depth (left) and comparision of different time stepping schemes(right).

The axisymmetric form of Richards' equation was set in an unsaturated flow region of sandy soil. The domain was specified as an annulus, $r_{\text{inner}} \leq r \leq r_{\text{outer}}$, with $r_{\text{inner}} > 0$. The annular domain is 70 cm in length and 100 cm in radius. A constant water head $\psi = -61.5$ cm was maintained at the bottom boundary, while a constant flux $q(t) = 13.69$ cm/hr for $t < 0.7$ hr and zero normal flux condition for $t > 0.7$ hr were applied at the soil surface.

The solution domain was meshed with a spatial step size $\Delta z = 2$ cm in the axial direction and $\Delta r = 2.5$ cm in the radial direction. The simulation was run for 1 hr. Figures 4 and 5 show the longitudinal and radial water content profiles and the corresponding surface plots.

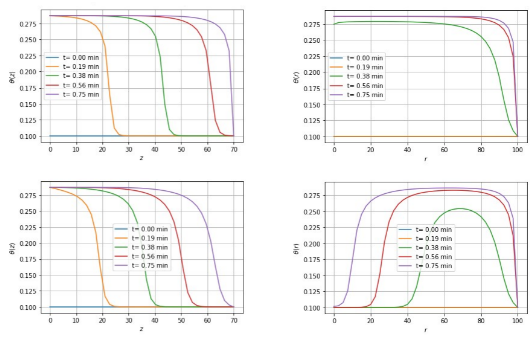

Figure 3: Longitudinal profile of soil moisture $\theta(z)$ at r= 50 cm(top left),radial profile of soil moisture $\theta(z)$ at z= 35 cm(top right), Longitudinal profile of soil moisture $\theta(z)$ at r= 25 cm(bottom left), radial profile of soil moisture $\theta(z)$ at z= 35 cm(bottom right)

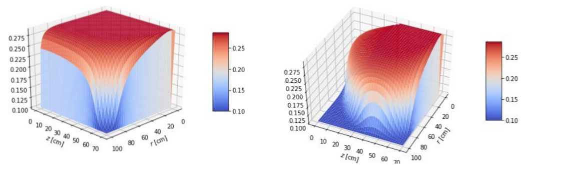

Figure 4: Soil Moisture $\theta(r, z)$ at $t = 0.75$sec for $r = R_{out}$(left), Soil Moisture $\theta(r, z)$ at $t = 0.75$sec for $r = R_{out}$ 2(right)

The safety factor for different pressure heads and corresponding moisture content is shown in Figure 6. The slope angle and thickness of the unsaturated soil were 30° and 8 m, respectively. The effective cohesion was 4.4 kPa, friction angle 28°, and soil unit weight 23.4 kN/m^3. The results indicate an increased likelihood of surface failure with heavy rainfall.

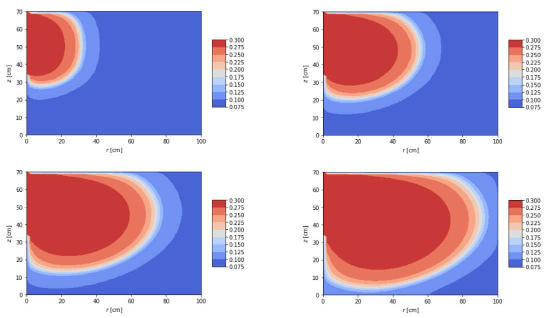

Figure 5: Soil Moisture $\theta(r, z)$ at $t = 0.1875$sec for $r = R_{out}$(top left), Soil Moisture $\theta(r, z)$ at $t = 0.375$sec for $r = R_{out}$(top right), Soil Moisture $\theta(r, z)$ at $t = 0.5625$sec for $r = R_{out}$(bottom left), Soil Moisture $\theta(r, z)$ at $t = 0.75$sec for $r = R_{out}$(bottom right)

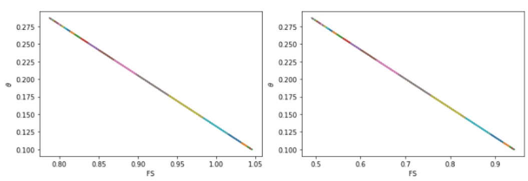

Figure 6: Factor of safety for different pressure head and the corresponding moisturity

Conclusion

The flow of water in unsaturated soil is described by axi symmetrical Kirchhoff trans- formed Richards equation (a highly nonlinear degenerate parabolic partial differential equation) and solved it numerically and interpolate the result in the hydrological model for stability analyses for the possibilities of landslide hazards. Also this work is based on illustrative numerical examples in cylindrical form, with realistic parameters. We implemented different finite difference schemes. The work presented here describes and verifies the application and accuracy of finite difference schemes to simulate flow in unsaturated porous media in axi symmetrical cylindrical formation of the land surface. However, this work presented a new super time stepping numerical scheme which was able to solve the Kirchhoff transformed axi symmetrical Richards equation and the findings are converges to the theoretical analysis and also the findings are in line with one dimensional approaches [42, 43]. Figures 2 and 3 depicted that the numerical simulations results for the same infiltration experiment as in one dimension imposing, zero flux boundary conditions in the lateral boundary mimic the results from the one dimensional model. That is the longitudinal profile are in line with that of the one dimensional model. The numerical method is able to handle short duration infiltration and relatively easy to implement. The interpolated result shows that the prediction of landslide is possible for the available parameter depending on the constituent of the soil and nature of the land. This work can be extended to achieve to accurate prediction of massive landslide to unsaturated heterogeneous soils with abruptly changing wetness conditions.

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Data Availability

The data supporting the findings of this study are included within the article.

Conflicts of Interest

The author declares no conflict of interest.

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