BIBECHANA
Vol. 21, No. 2, August 2024, 103-112

ISSN 2091-0762 (Print), 2382-5340 (Online)
Journal homepage: http://nepjol.info/index.php/BIBECHANA
Publisher:Dept. of Phys., Mahendra Morang A. M. Campus (Tribhuvan University)Biratnagar

Geometrical Interpretation of Space Contraction in Two-dimensional Lorentz Transformation

Chandra Bahadur Khadka

Department of Physics, Tri-Chandra Multiple Campus, Tribhuvan University, Kathmandu-44600, Nepal

*Corresponding author. Email: chandrabahadur9988@gmail.com

Abstract

This paper points out that the transformation equations for the spatial and temporal coordinates between two frames of reference in the existing generally accepted version of the Lorentz transformation are deficient, since transformation equations are based on one dimensional motion between inertial frames. Therefore, all possible space-time coordinate transformation equations between moving and stationary frames by prolonging Lorentz transformation in a two-dimensional plane are thoroughly proposed in this article. If v *denotes the relative velocity between stationary frame* (x, y) *and moving frame* $(x^{'}, y^{'})$*, then the transformation equations along X and Y-axis under two-dimensional Lorentz are given, respectively, by the formulas* $x^{'} = \frac{\left(x - \frac{vtx}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$ *and* $y^{'} = \frac{\left(y - \frac{vty}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$*. In this work we conclude that length and breadth of a rectangle appears to be shortened to the observer when there is the relative velocity between the rectangle and observer along both X and Y-axis. In particular, we present a concise and carefully reasoned account of a new aspect of Lorentz transformation which decently allows for the determination of space-time coordinates transformation equations in two dimensions of space.*

Keywords
Frame of reference, Lorentz transformation, Space contraction, Special relativity.

Article information
Manuscript received: September 5, 2023; Revised: January 10, 2024; Accepted: January 19, 2024
DOI https://doi.org/10.3126/bibechana.v21i2.62271

1 Introduction

The theory of special relativity [1, 2] at the current time stands as a universal theory comprising the relativistic nature of mass and the common space-time arena in which all fundamental phenomena occur. The well-known Lorentz transformation equations [3] form the basis of Einstein's special relativity and give the transformation equations for spatial and temporal coordinates in two frames of reference on the basis of constancy of light and its independence with relative velocity between source and observer.

There are many well- known written texts on the formulation of relativistic mechanics. The relativistic space-time coordinate formulas were presented in 1904 by Lorentz [3], and later that the Lorentz transformation was simplified and clearly recorded on 5th June 1905 by Poincare [4], who gave it a form very close to its contemporary version:

$$x^{'} = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, t^{'} = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y^{'} = y, z^{'} = z \tag{1}$$

where y, z,$y^{'}$ and $z^{'}$ are coordinates along the direction perpendicular to the direction of motion. On 30 June 1905, Einstein published his work [5] where based on the postulate of constant speed of light, he derived the correct transformation of coordinates, law for addition of velocities and mass energy equivalence principle. Lewis and Tolman [6] extended the relativity principle and introduced the theoretical dependence of mass on velocity. Derivations such as that by Poincare for space-time coordinates transformation between inertial frames and by Tolman for relativistic mass are propagated in several excellent textbooks, including famous Feynman's lectures [7]. There are some known research results about the extension of Lorentz transformation for determination of space time coordinate transformation along various directions. There are research papers that provide a deeper sense of the concept of relativity with the universal frame of reference [8] and all possible experiments to falsify such theories were conducted in [9]. In the article [10], new mathematical formalism of special relativity was developed. Research is also conducted on the practical aspects of relativity [11]. In Ref. [12], the trajectory of a relativistic particle in 1+1 dimensions was obtained in the representation of Lobachevsky geometry. The dynamics of a relativistic particle that does not have an electric charge and is under the action of an external force has been analyzed on the basis of the special theory of relativity in work [13]. Pagano et al [14] have discussed different roles of Lorentz transformation in classical wave propagation theories and in relativistic mechanics. Karplyuk et al [15] showed how the factorization of an arbitrary Lorentz transformation is performed as a sequence of a spatial rotation and a boost. Using the Lorentz transformation, Alex-Amor et al [16] had shown that a particular class of space-time modulated gratings behave effectively as moving media. Ref. [17] presents a concise and carefully reasoned account of a new aspect of gravitational redshift theory revealing the change of mass with gravity. Furthermore, Ref. [18] points out the possible extension of special relativity to derive the new mathematical formulas of linear momentum, force and kinetic energy. A recently published research article [19] shows the simultaneous contraction of length, breadth and height of cuboids by developing three-dimensional Lorentz transformation. There are many publications on special relativity with important theoretical results. The Lorentz transformation equations in most of the literature to date are formulated based on one-dimensional motion along X-axis between inertial frames governed by equation (1). Equation (1) generates the space contraction along the X-axis only and can't explain spatial behavior along Y and Z-directions. Therefore, it manifests the inadequacy of the current form of Lorentz transformation, especially the need for the inclusion of relative motion along both X and Y-directions between inertial frames of reference. Here, we present the corresponding modified Lorentz transformation in two dimensional XY plane by introducing the relative motion between inertial frames along both X and Y-directions simultaneously. The modified transformation equations, in place of equation (1) are:

$$x^{'} = \frac{x - \frac{vxt}{\sqrt{x^2+y^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}, y^{'} = \frac{y - \frac{vyt}{\sqrt{x^2+y^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t^{'} = \frac{t - \frac{v\sqrt{x^2+y^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, z^{'} = z \tag{2}$$

Comparing equations (1) and (2), the space coordinates transformation take place along only X-axis in old transformation (1) and time depends only on x coordinate while the space coordinates take place along both X-axis and Y-axis in modified transformation (2) and time depends on both x and y coordinates.

With above motivation, the structure of this paper is systematically arranged as follows. In section 2, the transformation equations along both X and Y-axis have been displayed by introducing the relative motion between inertial frames in the two-dimensional XY plane. In section 3, based on modified transformation equations, formulas of length and breadth contraction of a rectangle in a moving frame are obtained for the first time. Some conclusions are summed up in the last section.

2 Methods

2.1 The equation of wave front of light

Let two frame of reference s and s' such that s' frame of reference is moving with uniform velocity v as shown in figure (1). Let origin O and O' of two co-ordinate system coincide at $t = t' = 0$ and a source of light is flashed at the origin O at $t = 0$, when O and O' coincide. Then, in view of constancy of speed of electromagnetic wave with respect to the motion of two frames of reference, each observer at O and O' claim to be at the center of the spreading spherical wave front of light pulse. When light is at point P, let space-time co-ordinate measured by the observer O and O' be (x, y, z, t) and (x', y', z', t') respectively. Since both the observers are at the center of the same expanding wave front, the equation of wave front in frame s and frame s' must be equal.

$$x^2 + y^2 + z^2 - c^2t^2 = x^{'2} + y^{'2} + z^{'2} - c^2t^{'2} \quad (3)$$

Since there is no relative motion along Z-direction then,

$z = z^{'} = 0$

Above equation (3) reduces to

$$x^2 + y^2 - c^2t^2 = x^{'2} + y^{'2} - c^2t^{'2}$$

$$r^2 - c^2t^2 = r^{'2} - c^2t^{'2} \quad (4)$$

2.2 Geometrical calculations

In figure (1), θ be the angle made by line OP with positive X-axis in s frame of reference. Draw PQ' and PQ perpendiculars on X'-axis and X-axis respectively. In right angled triangles PQ'O' and PQO,

(i) angle OPQ = angle O'PQ' (same angle)

(ii) angle PQO = angle PQ'O' = 90°

(iii) angle QOP = angle Q'O'P (remaining angle)

Therefore, angle QOP = angle Q'O'P' = θ.

In right angled triangle O'PQ'

$$PO'^2 = O'Q'^2 + Q'P^2$$

or,

$$r^{'2} = x^{'2} + y^{'2} \quad (5)$$

Also,

$$sin\theta = \frac{PQ^{'}}{PO'} = \frac{y^{'}}{r'}$$

$$\cos\theta = \frac{Q^{'}O^{'}}{PO'} = \frac{x^{'}}{r'} \quad (6)$$

Similarly, in right angled triangle OPQ

$$PO^2 = OQ^2 + QP^2$$

$$, r^2 = x^2 + y^2 \quad (7)$$

Also,

$$sin\theta = \frac{PQ}{OP} = \frac{y}{r}$$

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} \quad (8)$$

2.3 Transformations Equations

In two dimensions, there is relative motion between inertial frames along both X and Y- directions simultaneously. Let r and r' be position of point P measured from frame of reference s and s' respectively. In figure (1), frame s' moves with velocity v in XY plane making an angle θ with positive X-axis of stationary frame. Therefore, velocity v has two components $v_x = v\cos\theta$ and $v_y = v\sin\theta$ along X-axis and Y-axis respectively.

Thus, the transformation equation from frame s to s' are

$$y^{'} = \gamma(y - v_yt)$$

$$x^{'} = \gamma(x - v_xt)$$

where $v_y = v\sin\theta$ and $v_x = v\cos\theta$ be component of velocity along Y and X-axis in s frame of reference respectively, then above transformation equation becomes,

$$y^{'} = \gamma(y - vtsin\theta)$$

$$x^{'} = \gamma(x - vtcos\theta)$$

where γ denotes the Lorentz factor. Now, we have,

$$y^{'} = \gamma\frac{(y - vtsin\theta)}{r} \quad (9)$$

$$x^{'} = \gamma\frac{(x - vtcos\theta)}{r} \quad (10)$$

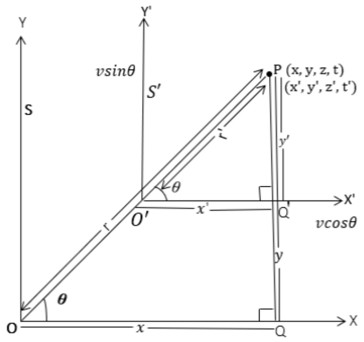

Figure 1: The inertial system s and s' in two dimensional planes

Now, radius vector r' in s' frame of reference from equation (5) is
$r'^2 = x'^2 + y'^2$

Substituting value from transformation equations (9) and (10),

$r'^2 = \gamma^2 \left(x - \frac{vxt}{r}\right)^2 + \gamma^2 \left(y - \frac{vyt}{r}\right)^2$

or,
$r'^2 = \gamma^2 \left(x^2 - \frac{2vtx^2}{r} + \frac{v^2x^2t^2}{r^2} + y^2 - \frac{2vty^2}{r} + \frac{v^2y^2t^2}{r^2}\right)$

or,
$r'^2 = \gamma^2 \left(x^2 + y^2 - 2tv\left(\frac{x^2+y^2}{r}\right) + t^2v^2\left(\frac{x^2+y^2}{r^2}\right)\right)$

Using equation (7) we get,

or, $r'^2 = \gamma^2 \left(r^2 - \frac{2vtr^2}{r} + \frac{t^2v^2r^2}{r^2}\right)$

or, $r'^2 = \gamma^2 \left(r^2 - 2tvr + t^2v^2\right)$

or, $r'^2 = \gamma^2 (r - vt)^2$

$$r' = \gamma(r - vt) \tag{11}$$

This is the transformation equation from frame s to frame s' in term of radius vector when there is relative motion along X and Y-axis simultaneously.

Again, the transformation equations from frame s' to s are

$$y = \gamma'(y'' + v'_y t')$$

$$x = \gamma'(x'' + x'_y t')$$

where $v'_y = vsin\theta$ and $v'_x = vcos\theta$ be component of velocity along Y and X- axis respectively, then above transformation equations becomes,

$$y = \gamma'(y'' + vsin\theta t')$$

$$x = \gamma'(x'' + vcos\theta t')$$

Substituting value of $sin\theta$ and $cos\theta$ from equation (6) we get,

$$y = \gamma'\left(y' + \frac{vy't'}{r'}\right) \tag{12}$$

$$x = \gamma'\left(x' + \frac{vx't'}{r'}\right) \tag{13}$$

Now, radius vector in s frame of reference from equation (7) is

$r^2 = x^2 + y^2$

Substituting value from transformation equation (12) and (13),

$r^2 = \gamma'^2 \left(x' + \frac{vx't'}{r'}\right)^2 + \gamma'^2 \left(y' + \frac{vy't'}{r'}\right)^2$

or, $r^2 = \gamma'^2 \left(x^2 + \frac{2vt'x'^2}{r'} + \frac{v^2x'^2t'^2}{r'^2} + y'^2 + \frac{2vt'y'^2}{r'} + \frac{v^2y'^2t'^2}{r'^2}\right)$

or, $r^2 = \gamma'^2 \left(x'^2 + y'^2 + 2vt'\left(\frac{x'^2+y'^2}{r'}\right) + v^2t'^2\left(\frac{x'^2+y'^2}{r'^2}\right)\right)$

Using equation (5) we get,

or, $r^2 = \gamma'^2 \left(r'^2 + \frac{2vt'r'^2}{r'} + \frac{v^2t'^2r'^2}{r'^2}\right)$

or, $r^2 = \gamma'^2 \left(r'^2 + 2vt'r' + v^2t'^2\right)$

or, $r^2 = \gamma'^2 \left(r' + vt'\right)^2$

or,

$$r = \gamma'(r' + vt') \tag{14}$$

This is the transformation equation from frame s' to frame s in term of radius vector when there is relative motion along X and Y-axis simultaneously.

2.4 The Lorentz transformation equations

The transformation equation relating r' and r can be written from equation (11) and (14) as,

$r' = \gamma(r - vt)$

and $r = \gamma'\left(r' + vt'\right)$

or, $r = \gamma'\left(\gamma(r - vt) + vt'\right)$

or, $r = \gamma'\gamma r - \gamma'\gamma vt + \gamma' vt'$

or, $t' = \frac{r}{\gamma' v} - \frac{\gamma'\gamma r}{\gamma' v} + \frac{\gamma'\gamma vt}{\gamma' v}$

or, $t' = \gamma t + \frac{r}{\gamma' v} - \frac{\gamma r}{v}$

or,

$$t' = \gamma\left[t - \frac{r}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right] \tag{15}$$

Equation (3) gives equation of wave front,

or, $r^2 - c^2t^2 = r'^2 - c'^2t'^2$

Substituting value of r' and t' from (11) and (15), we get

$r^2 - c^2t^2 = \gamma^2(r - vt)^2 - c^2\gamma^2\left[t - \frac{r}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right]^2$

or, $r^2 - c^2t^2 = \gamma^2 r^2 - 2\gamma^2 rvt + \gamma^2 v^2 t^2 - c^2\gamma^2 t^2 + 2c^2\gamma^2 t\frac{r}{v}\left(1 - \frac{1}{\gamma\gamma'}\right) - c^2\gamma^2\frac{r^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2$

or, $r^2 - c^2t^2 = r^2\left[\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2\right] + rt\left[-2\gamma^2 v + \frac{2c^2\gamma^2}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right] + t^2\left(\gamma^2 v^2 - c^2\gamma^2\right)$

Equating the Coefficients of r^2 ,rt and t^2 on both sides,

$$\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1 - \frac{1}{\gamma\gamma'}\right)^2 = 1 \tag{16}$$

$$-2\gamma^2 v + \frac{2c^2\gamma^2}{v}\left(1 - \frac{1}{\gamma\gamma'}\right) = 0 \tag{17}$$

$$\gamma^2 v^2 - \gamma^2 c^2 = -c^2 \tag{18}$$

From equation (18)

$-\gamma^2\left(c^2 - v^2\right) = -c^2$

$\gamma^2 = \frac{c^2}{(c^2 - v^2)} = \frac{1}{1 - \frac{v^2}{c^2}}$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{19}$$

Further, $\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$

or, $1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$

or,

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \tag{20}$$

From equation (17), we have,

$-v + \frac{c^2}{v}\left(1 - \frac{1}{\gamma\gamma'}\right) = 0$

or, $\frac{-v^2 + c^2\left(1 - \frac{1}{\gamma\gamma'}\right)}{v}$

or, $v^2 = c^2\left(1 - \frac{1}{\gamma\gamma'}\right)$

or,

$$\frac{v^2}{c^2} = \left(1 - \frac{1}{\gamma\gamma'}\right) \tag{21}$$

Comparing equations (20) and (21) we get,

$1 - \frac{1}{\gamma^2} = \left(1 - \frac{1}{\gamma\gamma'}\right)$

or, $\frac{1}{\gamma} = \frac{1}{\gamma'}$

or, $\gamma = \gamma' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Therefore, the required transformation equation from (11) is

$r' = \frac{r - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$

or,

$$\sqrt{x'^2 + y'^2} = \frac{\sqrt{x^2 + y^2} - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{22}$$

From (15) we get,

$t' = \gamma\left[t - \frac{r}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right]$

Using equation (21) we get,

or, $t' \; \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left[t - \frac{r}{v}.\frac{v^2}{c^2}\right]$

or, $t' = \frac{t - \frac{rv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$

or, $t' = \frac{t - v\frac{\sqrt{x^2 + y^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$

The inverse Lorentz transformation equations are obtained by interchanging the coordinates and changing v by $-v$ in above transformation equations as follows.

$r = \frac{r' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$

or, $\sqrt{x^2 + y^2} = \frac{\sqrt{x'^2 + y'^2} + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$t = \frac{t' + v\frac{\sqrt{x'^2 + y'^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{23}$$

These equations convert measurement made in frame s' into those in frame s. From these equations it is thus amply clear that space time coordinates transformation equations in two-dimensional space (XY plane) depend upon both x and y coordinates.

3 Results and Discussions

The transformation equation from moving frame to stationary frame of reference when there is relative motion along X and Y-axis can be written from equations (12) and (13) as follows.

$$y = \gamma'\left(y' + \frac{vy't'}{r'}\right) = \frac{y' + \frac{vy't'}{r'}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24}$$

$$x = \gamma'\left(x' + \frac{vx't'}{r'}\right) = \frac{x' + \frac{vx't'}{r'}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{25}$$

Above equations can be modified by using equation (6) as follows,

$y = \gamma'\left(y'' + vt'sin\theta\right) = \frac{y' + vt'sin\theta}{\sqrt{1 - \frac{v^2}{c^2}}}$

$x = \gamma'\left(x'' + vt'cos\theta\right) = \frac{x' + vt'cos\theta}{\sqrt{1 - \frac{v^2}{c^2}}}$

Now, radius vector in S frame of reference from equation (7) is

$r = \sqrt{x^2 + y^2}$

Using equations (24) and (25) we get,

or, $r = \sqrt{\left(\frac{x' + \frac{vx't'}{r'}}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2 + \left(\frac{y' + \frac{vy't'}{r'}}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2}$

or, $r = \sqrt{\frac{\left(x' + \frac{vx't'}{r'}\right)^2 + \left(y' + \frac{vy't'}{r'}\right)^2}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^2}}$

or, $r = \frac{\sqrt{x^2 + \frac{2vt'x'^2}{r'} + \frac{v^2x'^2t'^2}{r'^2} + y'^2 + \frac{2vt'y'^2}{r'} + \frac{v^2y'^2t'^2}{r'^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}$

or, $r = \frac{\sqrt{x'^2 + y'^2 + 2vt'\left(\frac{x'^2 + y'^2}{r'}\right) + v^2t'^2\left(\frac{x'^2 + y'^2}{r'^2}\right)}}{\sqrt{1 - \frac{v^2}{c^2}}}$

Substituting the value $r^{'2} = x^{'2} + y^{'2}$ we get,

or, $r = \frac{\sqrt{r'^2+2vt'\left(\frac{r'^2}{r'}\right)+v^2t'^2\left(\frac{r'^2}{r'^2}\right)}}{\sqrt{1-\frac{v^2}{c^2}}}$

or, $r = \frac{\sqrt{r'^2+2vt'r'+v^2t'^2}}{\sqrt{1-\frac{v^2}{c^2}}}$

or, $r = \frac{\sqrt{\left(r'+vt'\right)^2}}{\sqrt{1-\frac{v^2}{c^2}}}$

or,

$$r = \frac{r^{'} + vt^{'}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{26}$$

This is the required transformation equation in terms of radius vector. This process of mathematical calculation reveals that transformation equations (24) and (25) in terms of x and y coordinates are separated form of transformation equation (26).

Putting value of r and$r^{'}$ on equation (26) we get,

$$\sqrt{x^2+y^2} = \frac{\sqrt{x'^2+y'^2} + vt^{'}}{\sqrt{1-\frac{V^2}{c^2}}} \tag{27}$$

Case(I): when there is relative motion along single dimension say X-axis, then $y = y^{'} = 0$. As a result, equation (27) becomes,

$\sqrt{x^2} = \frac{\sqrt{x'^2}+vt^{'}}{\sqrt{1-\frac{V^2}{c^2}}}$

or,

$$x = \frac{x^{'} + vt^{'}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{28}$$

Which is the desired ordinary Lorentz transformation equation of space coordinate along X-axis.

Case (II): when there is relative motion along single dimension say Y-axis, then $x = x^{'} = 0$. As a result, equation (27) becomes,

$\sqrt{y^2} = \frac{\sqrt{y'^2}+vt^{'}}{\sqrt{1-\frac{V^2}{c^2}}}$

or,

$$y = \frac{y^{'} + vt^{'}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{29}$$

Which is the desired ordinary Lorentz transformation equation of space coordinate along Y-axis.

Consider two frames of references s and $s^{'}$ such that $s^{'}$ frame is moving with velocity v in two dimensional XY plane as shown in figure (2). Consider a rectangle ABCD in stationary frame of reference. Let (x_1, y_1) and (x_2, y_2) are coordinates of point A and C measured from s frame while $\left(x_1{'}, y_1{'}\right)$ and $\left(x_2{'}, y_2{'}\right)$ are coordinates of point A and C measured from $s^{'}$ as shown in figure (2).

Let length and breadth of rectangle are taken along X-axis and Y-axis respectively, then the length of the rectangle along AB in figure (2) is given by,

$$L_0 = x_2 - x_1 \tag{30}$$

This length L_0 of the rectangle in stationary frame of reference is known as proper length. The inverse Lorentz transformation equation along X-axis can be written from equation (25) as follows.

$x_1 = \frac{x_1{'} + \frac{vx_1{'}t^{'}}{r_1{'}}}{\sqrt{1-\frac{v^2}{c^2}}}$

$x_2 = \frac{x_2{'} + \frac{vx_2{'}t^{'}}{r_2{'}}}{\sqrt{1-\frac{v^2}{c^2}}}$

Putting these values in equation (30) we get,

$L_0 = \frac{x_2{'} + \frac{vx_2{'}t^{'}}{r_2{'}}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1{'} + \frac{vx_1{'}t^{'}}{r_1{'}}}{\sqrt{1-\frac{v^2}{c^2}}}$

or, $L_0 = \frac{x_2{'} - x_1{'} + \frac{vx_2{'}t^{'}}{r_2{'}} - \frac{vx_1{'}t^{'}}{r_1{'}}}{\sqrt{1-\frac{v^2}{c^2}}}$

or, $L_0\sqrt{1-\frac{v^2}{c^2}} = x_2{'} - x_1{'} + \frac{vx_2{'}t^{'}}{r_2{'}} - \frac{vx_1{'}t^{'}}{r_1{'}}$

or, $x_2{'} - x_1{'} = L_0\sqrt{1-\frac{v^2}{c^2}} + \frac{vx_1{'}t^{'}}{r_1{'}} - \frac{vx_2{'}t^{'}}{r_2{'}}$

or, $x_2{'} - x_1{'} = L_0\sqrt{1-\frac{v^2}{c^2}} + vt^{'}\left(\frac{x_1{'}}{r_1{'}} - \frac{x_2{'}}{r_2{'}}\right)$

or,

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} + vt^{'}\left(\frac{x_1{'}}{\sqrt{x'^2_1+y'^2_1}} - \frac{x_2{'}}{\sqrt{x'^2_2+y'^2_2}}\right) \tag{31}$$

Where $L = x_2{'} - x_1{'}$ be the length of rectangle along X-axis in frame $s^{'}$ called improper length, $r_1{'} = \sqrt{x'^2_1+y'^2_1}$ be radius vector measured from$s^{'}$of point A having coordinate $\left(x_1{'}, y_1{'}\right)$and $r_2{'} = \sqrt{x'^2_2+y'^2_2}$ be radius vector measured from$s^{'}$ of point C having coordinate $\left(x_2{'}, y_2{'}\right)$.

Similarly, breath of the rectangle along BC in figure (2) is given by,

$$B_0 = y_2 - y_1 \tag{32}$$

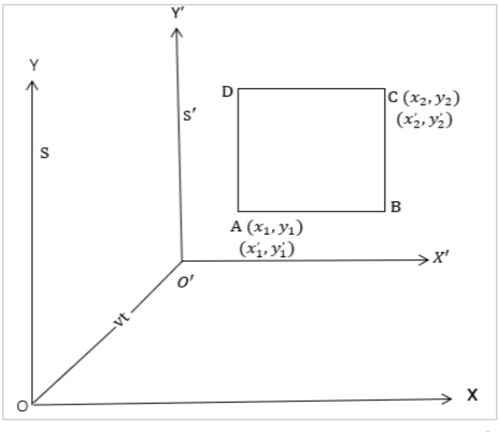

Figure 2: Length and breadth contraction of the rectangle

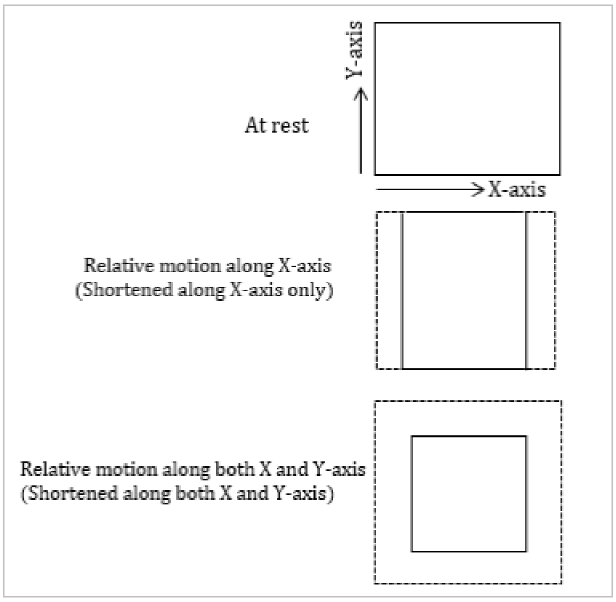

Figure 3: Relative geometrical shape of rectangle

Table 1: Space-time coordinates transformation.

S.N.	Direction of motion	Space coordinate transformation		Transformation equation of time
		Along X- axis	Along Y- axis	
1	Along both X and Y-axis	From equation (25), $x=\frac{x'+\frac{vx't'}{\sqrt{x'^2+y'^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$	From equation (24), $y=\frac{y'+\frac{vy't'}{\sqrt{x'^2+y'^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$	From equation (23), $t=\frac{t'+v\frac{\sqrt{x'^2+y'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ (Time depends upon both x' and y' coordinates.)
2	Along X-axis only $y=y'=0$	$x=\frac{x'+\frac{vx't'}{\sqrt{x'^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $x=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$	$y=\frac{0+\frac{v0t'}{\sqrt{x'^2+y'^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $y=0$	$t=\frac{t'+v\frac{\sqrt{x'^2+0^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ $t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$
3	Along Y-axis only $x=x'=0$	$x=\frac{0+\frac{v0t'}{\sqrt{x'^2+y'^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $x=0$ (No space coordinate transformation along X-axis)	$y=\frac{y'+\frac{vy't'}{\sqrt{y'^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $y=\frac{y'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$ (Space coordinate transformations take place along Y-axis only.)	$t=\frac{t'+v\frac{\sqrt{0^2+y'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ $t=\frac{t'+\frac{vy'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ (Time depends only upon y' coordinate.)

This breadth B_0 of the rectangle in stationary frame of reference is known as proper breadth. The inverse Lorentz transformation equation along Y-axis can be written from equation (24) as follows.

$$y_1=\frac{y_1{}'+\frac{vy_1{}'t'}{r_1{}'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y_2=\frac{y_2{}'+\frac{vy_2{}'t'}{r_2{}'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Putting these values in equation (32) we get,

$$B_0=\frac{y_2{}'+\frac{vy_2{}'t'}{r_2{}'}}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{y_1{}'+\frac{vy_1{}'t'}{r_1{}'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{or, } B_0=\frac{y_2{}'-y_1{}'+\frac{vy_2{}'t'}{r_2{}'}-\frac{vy_1{}'t'}{r_1{}'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{or, } B_0\sqrt{1-\frac{v^2}{c^2}}=y_2{}'-y_1{}'+\frac{vy_2{}'t'}{r_2{}'}-\frac{vy_1{}'t'}{r_1{}'}$$

$$\text{or, } y_2{}'-y_1{}'=B_0\sqrt{1-\frac{v^2}{c^2}}+\frac{vy_1{}'t'}{r_1{}'}-\frac{vy_2{}'t'}{r_2{}'}$$

$$\text{or, } y_2{}'-y_1{}'=B_0\sqrt{1-\frac{v^2}{c^2}}+vt'\left(\frac{y_1{}'}{r_1{}'}-\frac{y_2{}'}{r_2{}'}\right)$$

or,

$$B=B_0\sqrt{1-\frac{v^2}{c^2}}+vt'\left(\frac{y_1{}'}{\sqrt{x'^2_1+y'^2_1}}-\frac{y_2{}'}{\sqrt{x'^2_2+y'^2_2}}\right) \tag{33}$$

where $B=y_2{}'-y_1{}'$ be the breadth of rectangle along Y-axis in frame s' called improper breadth, $r_1{}'=\sqrt{x'^2_1+y'^2_1}$ be radius vector measured froms' of point A having coordinate $\left(x_1{}',y_1{}'\right)$ and

$r_2{}' = \sqrt{x'^2_2 + y'^2_2}$ be radius vector measured from$s'$of point C having coordinate $\left(x_2{}', y_2{}'\right)$.

From equations (31) and (33), it is seen that both length and breadth of rectangle appear to be shortened to observer in frames' if there is relative motion between rectangle and observer along both X-axis and Y-axis. On the other hand, the length of the rectangle appears to be shortened keeping breadth fixed if there is relative motion along the X-axis only. Similarly, the breadth of the rectangle appears to be shortened keeping length fixed if there is relative motion along Y-axis only. A neat depict of space contraction in one and two-dimensional motion is shown in figure (3).

Figure (3) shows that a square in one stationary frame appears to the observer in the other frame in one dimensional motion (say along X-axis) to be rectangle due to contraction along X-axis. A square gets shortened in both X and Y-directions if there is relative motion along both X and Y-axis simultaneously as shown in figure (3). The transformation equations along the X and Y-axis that give rise to such phenomena of shortening in spatial coordinates have been meticulously presented in table (1).

From the table, it is clearly seen that the modified Lorentz transformation represents the generalization of ordinary transformation (one-dimension system). Therefore, the modified transformation is the description of the behavior of the space-time coordinate transformation happening in two dimensions of space.

4 Conclusions

All possible space-time coordinate transformation equations by introducing relative motion between inertial frames in two dimensional XY-plane have been thoroughly outlined in this article. On the basis of transformation equations, it is definitely possible to obtain space contraction along the X and Y-axis simultaneously. The corresponding length and breadth contraction of rectangle when there is relative motion along both X and Y-axis can be written from equation (31) and (33) as follows.

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} + vt'\left(\frac{x_1{}'}{\sqrt{x'^2_1+y'^2_1}} - \frac{x_2{}'}{\sqrt{x'^2_2+y'^2_2}}\right)$$

$$B = B_0\sqrt{1 - \frac{v^2}{c^2}} + vt'\left(\frac{y_1{}'}{\sqrt{x'^2_1+y'^2_1}} - \frac{y_2{}'}{\sqrt{x'^2_2+y'^2_2}}\right)$$

From these equations, it should be concluded that both length and breadth of a rectangle appear to be shortened when there is relative motion along both X and Y-directions. Also, the transformation equation for temporal coordinate can be written from equation (23) as follows,

$$t' = \frac{t - v\frac{\sqrt{x^2+y^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

This equation reveals that time depends upon both x and y coordinate when there is relative motion along both X and Y- axis. When moving frames' in figure (1) moves along X-axis only, then $y = y' = 0$, hence above equation reduces to,

$$t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Which is the required transformation equation of time in ordinary Lorentz transformation (one dimensional system along X-axis). Hence, modified transformation is just an extended form of ordinary transformation in a two-dimensional plane. An extensive analysis of transformation of temporal and spatial coordinates between inertial frames can be obtained via the natural extension of Lorentz transformation in two dimensional spaces. The mathematical relations illuminated here will have many applications for manifestation of structure of space-time coordinate transformation between inertial frames of reference and will play an important role in many other areas of theoretical physics, especially those which are connected with relativistic mechanics.

Conflict of interest

The author declares no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial or not for profit sectors.

References

1. A. Einstein, Relativity: The special and the general theory-100th anniversary edition, Princeton University Press, Princeton, New Jersey, 2019.
2. R. Szostek. Derivation of numerous dynamics in the special theory of relativity. *Open Physics,* 17(1):157-166, 2019.
3. H. A. Lorentz. Electromagnetic phenomenon in a system moving with any velocity smaller than the that of light. *Proceeding of the Royal Netherlands Academy of arts and Sciences,* 6:809-831, 1904.
4. H. Poincare. On the dynamics of the electron. *In proc. Academic sciences,* 140:1504-1508, 1905.
5. A. Einstein. On the Electrodynamics of the moving bodies. *Annalen Der Physik,* 17(10):891-921, 1905.
6. G. N. Lewis and R. C. Tolman. *The principle of relativity and non-Newtonian mechanics. In proceedings of the American Academy of Arts and Sciences,* 44:709-726, 1909.

7. R. Feynman, R. B. Leighton and M. Sands, The Feynman lectures on physics, vol.1: The new millennium edition; mainly mechanics, radiation and heat.vol.1. Basic books, 2011.

8. R. Szostek. Explanation of what time of kinematics is and dispelling myths allegedly stemming from the special theory of relativity. *Applied Sciences,* 12(12):1-19, 2022.

9. K. Szostek and R. Szostek. The concept of mechanical for measuring the one-way speed of light. *Technical Transactions,* 2023:1-9, 2023.

10. G. M. Koczan. New definitions of 3D acceleration and inertial frames not violating F = MA in the special relativity. *Results in Physics,* 24:104121, 2021.

11. G. F. Sergery. The electromagnetic field outside the steadily rotating relativistic uniform system. *Jordan Journal of Physics,* 14(5):379-408, 2021.

12. N. S. Akintsov, A. P. Nevecheria and G. F. Kopytov. Modulated laser-induced acceleration of a relativistic charged particle. *Indian Journal of Physics,* 1–15, 2023.

13. N. S. Akintsov, A. P. Nevecheria and A. A. Martynov. The motion of an uncharged relativistic particle: an analysis of its integrable motion integrals dynamics. *St. Petersburg State Polytechnical University Journal. Physics and Mathematics,* 16 (2):132–145, 2023.

14. A. Pagano, and E. V. Pagano. A note on Lorentz transformation and simultaneity in classical physics and special relativity. *The European Physical Journal H*, 44(5):321-330, 2019.

15. K. Karplyuk, M. Kozak and O. Zhmudskyy. Factorization of the Lorentz transformations. *Ukrainian Journal of Physics*, 68(1):19-24, 2023.

16. A. Alex-Amor, C. Molero and M. G. Silveirinha. Analysis of Metallic Spacetime Gratings using Lorentz Transformations. *Physical Review Applied*, 20(1):014063, 2023.

17. C.B. Khadka. Determination of variation of mass with gravity. *Journal of Nepal Physical Society,* 9(1):129-136, 2023.

18. C.B. Khadka. An accurate theoretical formula for linear momentum, force and Kinetic energy. *BIBECHANA,* 20(3):259-266, 2023.

19. C.B. Khadka. Derivation of the Lorentz transformation for determination of space contraction. *St. Petersburg State Polytechnical University Journal. Physics and Mathematics,* 16(3):115-130, 2023.