**BIBECHANA**
A Multidisciplinary Journal of Science, Technology and Mathematics
ISSN 2091-0762 (Print), 2382-5340 (0nline)
Journal homepage: http://nepjol.info/index.php/BIBECHANA
**Publisher: Research Council of Science and Technology, Biratnagar, Nepal**

# On some curvature tensors in N(k)-contact metric manifolds

**Riddhi Jung Shah**
Department of Mathematics, Janata Campus, Nepal Sanskrit University, Dang, Nepal
Email: shahrjgeo@gmail.com

***Article history: Received 25 April, 2018; Accepted 14 September, 2018***
DOI: http://dx.doi.org/10.3126/bibechana.v16i0.19674



**Abstract**

The purpose of this paper is to study $W_7$ and $W_9$-curvature tensors on $N(k)$-contact metric manifolds. We prove that a $N(k)$-contact metric manifold satisfying the condition $W_7(\xi, X).W_9 = 0$ is $\eta$-Einstein manifold. We also obtain the Ricci tensor $S$ of type (0, 2) for $\varphi - W_9$ flat and $divW_9 = 0$ conditions on $N(k)$-contact metric manifolds. Finally, we give an example of 3-dimensional $N(k)$-contact metric manifold.

**Keywords:** Contact manifold; $N(k)$-contact metric manifold; $\eta$-Einstein manifold.

## 1. Introduction

In 1988, Tanno [1] introduced the notion of $k$-nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field or Reeb vector field (Reeb 1952) [2] $\xi$ of the contact metric manifold belongs to the distribution. A contact metric manifold with $\xi$ belonging to the $k$-nullity distribution is called $N(k)$-contact metric manifold. Generalizing this notion Blair, Koufogiorgos and Popantoniou [3] introduced the notion of a contact metric manifold with $\xi$ belonging to the $(k, \mu)$-nullity distribution, where $k$ and $\mu$ are real constants. In particular, if $\mu = 0$, then the notion of $(k, \mu)$-nullity distribution reduces to the notion of $k$-nullity distribution.

On the other hand, in 1982, Pokhariyal [4] defined $W_7$, $W_8$ and $W_9$-curvature tensors with the help of the Weyl's projective curvature tensor defined by Eisenhart [5]. In [4] it is studied that distribution of vector field over the metric potentials and matter tensors plays an important role in shaping the various physical and geometrical properties of a tensor. In the same paper it is proved that $W_9$-curvature tensor satisfies the cyclic property.

In this paper, we study some curvature conditions of $W_7$ and $W_9$-curvature tensors in $N(k)$-contact metric manifolds and obtain some results.

## 2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $M$ is said to be a contact manifold, if it carries a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form $\eta$, it is well known that there exists a unique vector field $\xi$, called the characteristic vector field or Reeb vector field (Reeb 1952) [2] of $\eta$, satisfying

$$\eta(\xi) = 1,\ d\eta(\xi, X) = 0 \tag{2.1}$$

for all vector fields $X$. A Riemannian metric $g$ is said to be an associated metric if there exists a tensor field $\varphi$ of type (1, 1) such that

$$d\eta(X,Y) = g(X, \varphi Y),\ \ \eta(X) = g(X, \xi),\ \ \varphi^2(X) = -X + \eta(X)\xi \tag{2.2}$$

for all vector fields $X, Y$ on $M^{2n+1}$.

From above conditions one can easily obtain

$$\varphi\xi = 0,\ \ \eta \circ \varphi = 0,\ \ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.3}$$

The structure $(\varphi, \xi, \eta, g)$ is called a contact metric structure and a manifold $M^{2n+1}$ with a contact metric structure $(\varphi, \xi, \eta, g)$ is said to be a contact metric manifold [6].

Now, we define the operators $l$ and $h$ by

$$lX = R(X, \xi)\xi,\ \ h = \frac{1}{2} \pounds_{\xi} \varphi \tag{2.4}$$

where $R$ and $\pounds$ denote the curvature tensor and Lie differentiation respectively. The (1, 1)-type tensors $h$ and $l$ are symmetric and satisfy

$$h\xi = 0,\ \ l\xi = 0,\ \ Tr.h = 0,\ \ Tr.h\varphi = 0 \text{ and } h\varphi = -\varphi h. \tag{2.5}$$

We also have the following relations for a contact metric manifold:

$$\nabla_X \xi = -\varphi X - \varphi h X \tag{2.6}$$

$$\nabla_{\xi} \varphi = 0 \tag{2.7}$$

$$Tr.l = g(Q\xi, \xi) = 2n - Tr.h^2 \tag{2.8}$$

$$\varphi l \varphi - l = 2(\varphi^2 + h^2) \tag{2.9}$$

$$\nabla_{\xi} h = \varphi - \varphi l - \varphi h^2 \tag{2.10}$$

where $Q$ is the Ricci operator defined by $g(QX,Y) = S(X,Y)$, $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$ and $S$ the Ricci tensor [6,7].

An almost contact metric manifold is an odd dimensional smooth manifold $M^{2n+1}$ equipped with an almost contact metric structure $(\varphi, \xi, \eta, g)$ satisfying the conditions (2.1) - (2.3) except the condition $d\eta(X,Y) = g(X, \varphi Y)$ or $d\eta(\xi, X) = 0$. Sasaki and Hatakeyama [8] defined an almost complex structure $J$ on $M^{2n+1} \times R$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right) \tag{2.11}$$

where $f$ is a smooth real valued function on $M^{2n+1} \times R$. An almost contact metric structure is said to be normal if $J$ is integrable. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X. \tag{2.12}$$

A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.13}$$

for all vector fields $X$ and $Y$ [6].

A contact metric manifold for which $\xi$ is a Killing vector is called a $K$-contact manifold. It is well known that a contact manifold is $K$-contact if and only if $h=0$. We note that a Sasakian manifold is $K$-contact, but the converse holds only if $\dim M=3$.
The $k$-nullity distribution [1] of a Riemannian manifold $(M,g)$ for a real number $k$ is a distribution

$$N(k): p\to N_p(k)=\{Z\in T_p(M): R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]\} \tag{2.14}$$

for any $X,Y\in T_p(M)$. If the characteristic vector field $\xi\in N(k)$, then the relation

$$R(X,Y)\xi=k[\eta(Y)X-\eta(X)Y] \tag{2.15}$$

hods. A contact metric manifold satisfying the relation (2.15) is called a $N(k)$-contact metric manifold. From (2.13) and (2.15) it follows that a $N(k)$-contact metric manifold is a Sasakian manifold if and only if $k=1$. If $k=0$ i.e., $R(X,Y)\xi=0$, then $N(k)$-contact metric manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$. If $k<1$, the scalar curvature is $r=2n\{2(n-1)+k\}$.
In a $(2n+1)$-dimensional $N(k)$-contact metric manifold $M$, the following relations hold:

$$h^2=(k-1)\varphi^2,\;\; k\le 1 \tag{2.16}$$

$$Q\xi=2nk\xi, S(X,\xi)=2nk\eta(X) \tag{2.17}$$

$$R(\xi,X)Y=k[g(X,Y)\xi-\eta(Y)X] \tag{2.18}$$

$$(\nabla_X\varphi)(Y)=g(X+hX,Y)\xi-\eta(Y)(X+hX) \tag{2.19}$$

$$Tr.h^2=2n(1-k) \tag{2.20}$$

$$Q\varphi-\varphi Q=4(n-1)h\varphi \tag{2.21}$$

$$S(\varphi X,\varphi Y)=S(X,Y)-2nk\eta(X)\eta(Y)-4(n-1)g(hX,Y) \tag{2.22}$$

$$S(\varphi X,Y)+S(X,\varphi Y)=4(n-1)g(\varphi X,hY) \tag{2.23}$$

$$\eta(R(X,Y)Z)=k[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \tag{2.24}$$

$$(\nabla_X\eta)(Y)=g(X+hX,\varphi Y) \tag{2.25}$$

for any vector fields $X,Y$ on $M^{2n+1}$ [1, 3].
In $n$-dimensional space $V_n$ Pokhariyal [4] defined the $W_7$ and $W_9$ curvature tensors as follows

$$W_7(X,Y,Z,T)=R(X,Y,Z,T)+\frac{1}{n-1}[g(Y,Z)Ric(X,T)-g(X,T)Ric(Y,Z)]$$

and

$$W_9(X,Y,Z,T)=R(X,Y,Z,T)+\frac{1}{n-1}[g(Z,T)Ric(X,Y)-g(Y,Z)Ric(X,T)]$$

respectively.
In $(2n+1)$-dimensional $N(k)$-contact metric manifold above two tensors can be written as

$$W_7(X,Y)Z=R(X,Y)Z+\frac{1}{2n}[g(Y,Z)QX-S(Y,Z)X] \tag{2.26}$$

and

$$W_9(X,Y)Z=R(X,Y)Z+\frac{1}{2n}[S(X,Y)Z-g(Y,Z)QX] \tag{2.27}$$

respectively, where $g(R(X,Y)Z,T)=R(X,Y,Z,T)$ for any vector fields $X,Y,Z,T$ on $M^{2n+1}$.
Now, using (2.17) and (2.18) in (2.26) and (2.27) we obtain

$$W_7(\xi,X)Y=k[2g(X,Y)\xi-\eta(Y)X]-\frac{1}{2n}S(X,Y)\xi \tag{2.28}$$

and

$$W_9(\xi, X)Y = k[\eta(X)Y - \eta(Y)X] \tag{2.29}$$

respectively.

From (2.27) we also have

$$\eta(W_9(X,Y)Z) = \frac{1}{2n}S(X,Y)\eta(Z) - kg(X,Z)\eta(Y) \tag{2.30}$$

$$\eta\big(W_9(\xi, X)Y\big) = 0 \tag{2.31}$$

$$\eta\big(W_9(X,\xi)Y\big) = k[\eta(X)\eta(Y) - g(X,Y)] \tag{2.32}$$

$$\eta\big(W_9(X,Y)\xi\big) = \frac{1}{2n}S(X,Y) - k\eta(X)\eta(Y). \tag{2.33}$$

**Definition 2.1.** A $N(k)$-contact metric manifold is said to be $\eta$-Einstein if its Ricci tensor $S$ of type (0, 2) is of the form

$$S(X,Y) = \mu_1 g(X,Y) + \mu_2\eta(X)\eta(Y) \tag{2.34}$$

where $\mu_1, \mu_2$ are smooth functions on the manifold.

**Definition 2.2.** A $N(k)$-contact metric manifold is said to be $\varphi$-$W_9$ flat if the condition

$$g\big(W_9(\varphi X, \varphi Y)\varphi Z, \varphi W\big) = 0$$

holds for any vector fields $X, Y, Z, W \in T_p(M)$.

## 3. Results and Discussion

In this section we obtain some results on $W_7$ and $W_9$ curvature tensors in $N(k)$-contact metric manifolds.

**Theorem 3.1.** Let $M$ be a $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying the curvature condition $W_7(\xi, X).W_9 = 0$, then either $M$ is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ for $n>1$ or $M$ is $\eta$-Einstein manifold.

**Proof.** Let us consider a $(2n+1)$-dimensional $N(k)$-contact metric manifold $M$ which satisfies the curvature condition

$$\big(W_7(\xi, X).W_9\big)(U,V)Z = 0$$

for any vector fields $X, U, V, Z$ on $M^{2n+1}$. By definition we have

$$\begin{aligned}\big(W_7(\xi, X).W_9\big)(U,V)Z &= W_7(\xi, X)W_9(U,V)Z - W_9\big(W_7(\xi, X)U, V\big)Z \\ &\quad - W_9\big(U, W_7(\xi, X)V\big)Z - W_9(U,V)W_7(\xi, X)Z\end{aligned}$$

$$\begin{aligned}\text{or,}\quad 0 &= W_7(\xi, X)W_9(U,V)Z - W_9\big(W_7(\xi, X)U, V\big)Z \\ &\quad - W_9\big(U, W_7(\xi, X)V\big)Z - W_9(U,V)W_7(\xi, X)Z.\end{aligned} \tag{3.1}$$

In view of (2.28) and (3.1) we get

$$\begin{aligned}0 &= 2kg\big(X, W_9(U,V)Z\big)\xi - k\eta\big(W_9(U,V)Z\big)X - \frac{1}{2n}S\big(X, W_9(U,V)Z\big)\xi \\ &\quad - 2kg(X,U)W_9(\xi,V)Z + k\eta(U)W_9(X,V)Z + \frac{1}{2n}S(X,U)W_9(\xi,V)Z \\ &\quad - 2kg(X,V)W_9(U,\xi)Z + k\eta(V)W_9(U,X)Z + \frac{1}{2n}S(X,V)W_9(U,\xi)Z \\ &\quad - 2kg(X,Z)W_9(U,V)\xi + k\eta(Z)W_9(U,V)X + \frac{1}{2n}S(X,Z)W_9(U,V)\xi.\end{aligned} \tag{3.2}$$

Taking inner product on both sides of (3.2) by $\xi$ and using (2.1) and (2.2) we obtain

$$\begin{aligned}0 &= 2kg\big(X, W_9(U,V)Z\big) - k\eta\big(W_9(U,V)Z\big)\eta(X) - \frac{1}{2n}S\big(X, W_9(U,V)Z\big)\\ &- 2kg(X,U)\eta\big(W_9(\xi,V)Z\big) + k\eta(U)\eta\big(W_9(X,V)Z\big) + \frac{1}{2n}S(X,U)\eta\big(W_9(\xi,V)Z\big)\\ &- 2kg(X,V)\eta\big(W_9(U,\xi)Z\big) + k\eta(V)\eta\big(W_9(U,X)Z\big) + \frac{1}{2n}S(X,V)\eta\big(W_9(U,\xi)Z\big)\\ &- 2kg(X,Z)\eta\big(W_9(U,V)\xi\big) + k\eta(Z)\eta\big(W_9(U,V)X\big) + \frac{1}{2n}S(X,Z)\eta\big(W_9(U,V)\xi\big).\end{aligned} \tag{3.3}$$

By the use of (2.30) - (2.33), (3.3) reduces to

$$\begin{aligned}0 &= 2kg(X, W_9(U,V)Z) - \frac{1}{2n}S(X, W_9(U,V)Z)\\ &+ \frac{k}{n}S(X,V)\{\eta(U)\eta(Z) - \frac{1}{2}g(U,Z)\} + kg(X,Z)\{k\eta(U)\eta(V)\\ &- \frac{1}{n}S(U,V)\} + 2k^2 g(X,V)\{g(U,Z) - \eta(U)\eta(Z)\} + k\{S(X,U)\\ &- kg(X,U)\}\eta(V)\eta(Z) + \frac{1}{4n^2}S(X,Z)S(U,V) - \frac{k}{2n}S(X,Z)\eta(U)\eta(V).\end{aligned} \tag{3.4}$$

Putting $U = \xi$ in (3.4) and using (2.1), (2.2), (2.17) and (2.29) we get

$$\begin{aligned}0 &= k[kg(X,Z)\eta(V) - 2kg(X,V)\eta(Z) - \frac{1}{2n}S(X,Z)\eta(V)\\ &+ \frac{1}{n}S(X,V)\eta(Z) + (2n-1)k\eta(V)\eta(X)\eta(Z)].\end{aligned} \tag{3.5}$$

From (3.5) it follows that either $k = 0$ or

$$\begin{aligned}&[kg(X,Z)\eta(V) - 2kg(X,V)\eta(Z) - \frac{1}{2n}S(X,Z)\eta(V)\\ &+ \frac{1}{n}S(X,V)\eta(Z) + (2n-1)k\eta(V)\eta(X)\eta(Z)] = 0.\end{aligned}$$

If $k = 0,$ then we know that $N(k)$-contact metric manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ [9].

If

$$\begin{aligned}&[kg(X,Z)\eta(V) - 2kg(X,V)\eta(Z) - \frac{1}{2n}S(X,Z)\eta(V)\\ &+ \frac{1}{n}S(X,V)\eta(Z) + (2n-1)k\eta(V)\eta(X)\eta(Z)] = 0,\end{aligned}$$

then replacing $Z$ by $\xi$ we get

$$S(X,V) = 2nkg(X,V) - n(2n-1)k\eta(X)\eta(V). \tag{3.6}$$

From (3.6) we obtain

$$S(X,V) = \lambda_1 g(X,V) + \lambda_2 \eta(X)\eta(V). \tag{3.7}$$

where $\lambda_1 = 2nk$ and $\lambda_2 = -n(2n-1)k.$

In view of (2.34) and (3.7) we conclude that $N(k)$-contact metric manifold $M^{2n+1}$ is $\eta$-Einstein manifold. This completes the proof of the theorem.

**Theorem 3.2.** In a $\varphi$-$W_9$ flat $N(k)$-contact metric manifold $M^{2n+1}$ the Ricci tensor $S$ and the scalar curvature $r$ are given by

$$S(Y,Z) = (r - 4n^2k)g(Y,Z) + \{2n(2n+1)k - r\}\eta(Y)\eta(Z) + 4(n-1)g(hY,Z)$$

and $r = 2n(2n+1)k$ respectively.

**Proof.** Let us consider a $(2n+1)$-dimensional $N(k)$-contact metric manifold $M$ which is $\varphi$-$W_9$ flat. Then we have

$$g(W_9(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \tag{3.8}$$

for any vector fields $X, Y, Z, W \in T_p(M)$.

In view of (2.27) we have

$$g(W_9(\varphi X, \varphi Y)\varphi Z, \varphi W) = g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) + \frac{1}{2n}[S(\varphi X, \varphi Y)g(\varphi Z, \varphi W) - S(\varphi X, \varphi W)g(\varphi Y, \varphi Z)]. \tag{3.9}$$

Using (2.3), (2.14), (2.22) and (3.8) in (3.9) we obtain

$$\begin{aligned} 0 = {} & kg(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - kg(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \\ & + \frac{1}{2n}[\{S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y)\}\{g(W,Z) - \eta(W)\eta(Z)\} \\ & - \{S(X,W) - 2nk\eta(X)\eta(W) - 4(n-1)g(hX,W)\}\{g(Y,Z) - \eta(Y)\eta(Z)\}] \end{aligned}$$

or,

$$\begin{aligned} 0 = {} & kg(Y,Z)g(X,W) - kg(X,W)\eta(Y)\eta(Z) - kg(X,Z)g(Y,W) \\ & + kg(Y,W)\eta(X)\eta(Z) + kg(X,Z)\eta(Y)\eta(W) + \frac{1}{2n}S(X,Y)g(W,Z) \\ & - kg(W,Z)\eta(X)\eta(Y) - \frac{2(n-1)}{n}g(W,Z)g(hX,Y) \\ & - \frac{1}{2n}S(X,Y)\eta(W)\eta(Z) + \frac{2(n-1)}{n}g(hX,Y)\eta(W)\eta(Z) \\ & - \frac{1}{2n}S(X,W)g(Y,Z) + \frac{2(n-1)}{n}g(Y,Z)g(hX,W) \\ & + \frac{1}{2n}S(X,W)\eta(Y)\eta(Z) - \frac{2(n-1)}{n}g(hX,W)\eta(Y)\eta(Z). \end{aligned} \tag{3.10}$$

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = W = e_i$ in (3.10) and taking summation over $i, 1 \le i \le 2n+1$, we get

$$S(Y,Z) = (r - 4n^2k)g(Y,Z) + \{2n(2n+1)k - r\}\eta(Y)\eta(Z) + 4(n-1)g(hY,Z) \tag{3.11}$$

by the use of (2.5) and symmetry property of $h$.

Again, taking an orthonormal frame field at any point of the manifold and contracting over $Y$ and $Z$ in (3.11) we obtain

$$r = 2n(2n+1)k. \tag{3.12}$$

In view of (3.11) and (3.12), the theorem is proved.

**Theorem 3.3.** If a $(2n+1)$-dimensional $N(k)$-contact metric manifold $M$ satisfies the condition $divW_9 = 0$, then the Ricci tensor $S$ is of the form

$$S(Y,Z) = 2nkg(Y,Z) + \frac{2n(2n-1)k}{2n+1}g(Y,hZ) + S(hZ,Y).$$

**Proof.** Let us consider a $(2n+1)$-dimensional $N(k)$-contact metric manifold $M$ which satisfies the condition

$$(divW_9)(X,Y)Z = 0 \tag{3.13}$$

where '*div*' denotes the divergence.

Taking inner product on both sides of (2.27) with $V$ we get

$$g(W_9(X,Y)Z,V) = g(R(X,Y)Z,V) + \frac{1}{2n}[S(X,Y)g(Z,V) - S(X,V)g(Y,Z)]. \quad (3.14)$$

Differentiating (3.14) covariantly along $U$ we have

$$\begin{aligned}&(\nabla_U \widetilde{W}_9)(X,Y,Z,V)\\ &= (\nabla_U \widetilde{R})(X,Y,Z,V) + \frac{1}{2n}[(\nabla_U S)(X,Y)g(Z,V) - (\nabla_U S)(X,V)g(Y,Z)]\end{aligned} \quad (3.15)$$

where $g(R(X,Y)Z,V) = \widetilde{R}(X,Y,Z,V)$ for all $X,Y,Z,V$.

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $U = V = e_i$ in (3.15) and taking summation over $i, 1 \le i \le 2n+1,$ we get

$$(divW_9)(X,Y)Z = (divR)(X,Y)Z + \frac{1}{2n}[(\nabla_Z S)(X,Y) - \frac{dr(X)}{2}g(Y,Z)]. \quad (3.16)$$

From Bianchi second identity we also know that

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \quad (3.17)$$

In view of (3.13), (3.16) and (3.17) we have

$$0 = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{2n}(\nabla_Z S)(X,Y) - \frac{1}{4n}g(Y,Z)dr(X). \quad (3.18)$$

Replacing $X$ by $\xi$ in (3.18) we obtain

$$(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z) + \frac{1}{2n}(\nabla_Z S)(\xi,Y) = 0 \quad (3.19)$$

since $dr(\xi) = 0$.

From $\pounds_\xi S = 0$ we get

$$\begin{aligned}(\nabla_\xi S)(Y,Z) &= -S(\nabla_Y \xi, Z) - S(\nabla_Z \xi, Y)\\ &= 4(n-1)g(\varphi Y, hZ) + S(\varphi hY, Z) + S(\varphi hZ, Y)\end{aligned}$$

by the use of (2.6) and (2.23).

Again,

$$\begin{aligned}(\nabla_Y S)(\xi,Z) &= \nabla_Y S(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z)\\ &= 2nk(\nabla_Y \eta)(Z) + S(\varphi Y, Z) + S(\varphi hY, Z)\\ &= 2nkg(Y,\varphi Z) + 2nkg(hY,\varphi Z) + S(\varphi Y, Z) + S(\varphi hY, Z).\end{aligned}$$

by the use of (2.6), (2.17) and (2.25).

Similarly, $(\nabla_Z S)(\xi,Y) = 2nkg(Z,\varphi Y) + 2nkg(hZ,\varphi Y) + S(\varphi Z, Y) + S(\varphi hZ, Y)$.

Using above relations in (3.19) we get

$$\begin{aligned}0 = &\{4(n-1)+k\}g(\varphi Y, hZ) + \frac{(2n+1)}{2n}S(\varphi hZ, Y) - 2nkg(Y,\varphi Z)\\ &- 2nkg(hY,\varphi Z) - S(\varphi Y, Z) + kg(Z,\varphi Y) + \frac{1}{2n}S(\varphi Z, Y).\end{aligned} \quad (3.20)$$

Replacing $Z$ by $\varphi Z$ in (3.20) and using (2.2), (2.3), (2.5), (2.17), (2.22) and symmetry property of $h$ we obtain

$$\begin{aligned}0 = &\{4(n-1)+k\}g(\varphi Y, h\varphi Z) + \frac{(2n+1)}{2n}S(\varphi h\varphi Z, Y) - 2nkg(Y,\varphi^2 Z)\\ &- 2nkg(hY,\varphi^2 Z) - S(\varphi Y,\varphi Z) + kg(\varphi Y,\varphi Z) + \frac{1}{2n}S(\varphi^2 Z, Y)\end{aligned}$$

$$\text{or, } S(Y,Z) = 2nkg(Y,Z) + \frac{2n(2n-1)k}{2n+1}g(Y,hZ) + S(hZ,Y). \quad (3.21)$$

This completes the proof of the theorem.

**4. Example of 3-dimensional $N(k)$-contact metric manifold**

Consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3 : x \neq 0\}$ where $(x, y, z)$ are the standard coordinates in $R^3$. Let $\{e_1, e_2, e_3\}$ be linearly independent global frame on $M$ defined by

$$e_1 = \frac{2}{x}\frac{\partial}{\partial y}, e_2 = 2\frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z} \text{ and } e_3 = \frac{\partial}{\partial z}.$$

Let $g$ be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let $\eta$ be the 1-form defined by $\eta(V) = g(V, e_3)$ for any $V \in \chi(M)$, the set of vector fields. Let $\varphi$ be the (1, 1) tensor field defined by $\varphi e_1 = e_2, \varphi e_2 = -e_1, \varphi e_3 = 0$. Then using the linearity of $\varphi$ and $g$ we have

$$\eta(e_3) = 1,\ \varphi^2 V = -V + \eta(V)e_3 \text{ and } g(\varphi U, \varphi V) = g(U,V) - \eta(U)\eta(V)$$

for any $U, V \in \chi(M)$. Moreover, $he_1 = -e_1, he_2 = e_2, he_3 = 0$. Thus for $e_3 = \xi$, the structure $(\varphi, \xi, \eta, g)$ defines a contact metric structure on $M$.

Now, from definition of Lie bracket we have

$$[e_1, e_2] = \frac{2}{x}\frac{\partial}{\partial y}\left(2\frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}\right) - \left(2\frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}\right)\left(\frac{2}{x}\frac{\partial}{\partial y}\right)$$
$$= \frac{4}{x^2}\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}$$
$$= \frac{2}{x}e_1 + 2e_3.$$

Similarly, we obtain $[e_1, e_3] = 0$ and $[e_2, e_3] = 2e_1$.

Now, we have Koszul's formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$
$$- g(Y, [X, Z]) + g(Z, [X, Y]).$$

Taking $e_3 = \xi$ and using Koszul formula we obtain

$$2g(\nabla_{e_1} e_2, e_1) = \frac{4}{x}g(e_1, e_1),\ 2g(\nabla_{e_1} e_2, e_2) = 0 = \frac{4}{x}g(e_1, e_2) \text{ and } 2g(\nabla_{e_1} e_2, e_3) = 0 = \frac{4}{x}g(e_1, e_3).$$

From above it follows that $g(\nabla_{e_1} e_2, X) = \frac{2}{x}g(e_1, X)$. Hence $\nabla_{e_1} e_2 = \frac{2}{x}e_1$.

Similarly we obtain other results and finally we have

$$\nabla_{e_1} e_1 = -\frac{2}{x}e_2,\ \nabla_{e_1} e_2 = \frac{2}{x}e_1,\ \nabla_{e2} e_1 = -2e_3,\ \nabla_{e2} e_3 = 2e_1,$$
$$\nabla_{e1} e_3 = \nabla_{e2} e_2 = \nabla_{e3} e_1 = \nabla_{e3} e_2 = \nabla_{e3} e_3 = 0.$$

From above one can easily seen that the conditions for $N(k)$-contact metric manifold are satisfied for $e_3 = \xi$ and $k = -\frac{4}{x}, x \neq 0$.

## 5. Conclusion

In this paper we have studied $W_7$ and $W_9$ curvature tensors on $N(k)$-contact metric manifolds. It is proved that a $(2n+1)$-dimensional $N(k)$-contact metric manifold satisfying the condition $W_7(\xi, X).W_9 = 0$ is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or is $\eta$-Einstein. It is also investigated that a $N(k)$-contact metric manifold satisfying the conditions $\varphi - W_9$ flat and $divW_9 = 0$ has the Ricci tensor $S$ of different forms.

## References

[1] S. Tanno, Ricci curvatures of contact Riemannian manifolds, Tohoku Math. J. 40 (1988) 441-448.
[2] G. Reeb, Sur certaines proprietes topologiques des trajectoires des systemes dynamiques, Memoires de 1Acad. Roy. de Beligique, Sci. Ser. 2 (1952) 1-62.
[3] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, Israel J. of Math. 19 (1995) 189-214.
[4] G. P. Pokhariyal, Relativistic significance of curvature tensors, Internat. J. Math. and Math. Sci. 5(1) (1982) 133-139.
[5] L. P. Eisenhart, Riemannian geometry, Princeton University Press, 1950.
[6] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, 509, Springer-Verlag, Berlin, 1976.
[7] D. E. Blair, J. N. Patnaik, Contact manifolds with characteristic vector field annihilated by the curvature, Bull. Inst. Math. Acad. Sinica 9 (1981) 533-545.
[8] S. Sasaki, Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tohoku Math. J. 13 (1961) 281-294.
[9] C. Baikoussis, T. Koufogiorgos, On a type of contact manifolds, J. of Geometry 46 (1993) 1-9.